Ministry of Higher Education
& Scientific Research
University of Anbar
College of Science
Department of Applied
Mathematics

lectures
Subject: Vector analysis.
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Stage: 2st.
The lecturer: Assist. Prof. Dr.
Ali Rashid Ibrahim

Line and planes in 3-space:

In this section we shall use the vectors to derive equations of lines and planes in 3-spase.

We shall then use these equations to solve some basic geometric problems.

Planes in 3-space (Finding the equations of the plan):

We know that in analytic geometry a line in 2-space can be specified by giving its slope and one of its points. Similarly, we can specify a plane in 3-space by giving its inclination(slope) specifying one of its points. Describe the inclination of a plane is to specify a nonzero vector, called **normal**, that is perpendicular to the plane.

If we want to find **the equation of the plane passing through the point** $P_0= (x_0, y_0, z_0)$ and having the nonzero vector $\mathbf{n}= \langle a, b, c\rangle$ as a normal as shown in the following figure (figure 49), we note that the plane consists precisely of those points $P= (x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$, that is,

$$\mathbf{n} \cdot \overrightarrow{P_0P}= 0 \quad \dots (1)$$

Since $\overrightarrow{P_0P}= \langle x - x_0, y - y_0, z - z_0\rangle$, therefore the equation (1) can be written as,

$$a(x - x_0) + b(y - y_0) + c(z - z_0)= 0 \quad \dots (2)$$

If we continue, $\quad ax - ax_0 + by - by_0 + cz - cz_0= 0$

$$ax + by + cz + \underline{(-ax_0 - by_0 - cz_0)} = 0 \text{ (let } -ax_0 - by_0 - cz_0= d)$$

Thus, we can rewrite (2) in this form,

$$ax + by + cz +d= 0 \quad \dots (3)$$

Where a, b, c, and d are constants, and a, b, and c are not all zero.
We call this the **point-normal** form of the equation of a plane.

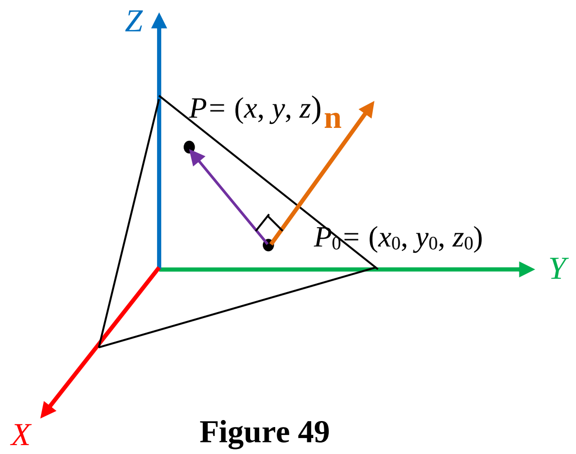

Figure 49

Plane with normal vector

Example (61): (Finding the point-normal equation of a plane).

Find the equation of the plane passing through the point (3, −1, 7) and perpendicular to the vector **n**= (4, 2, −5).

Solution:

By the formula (2),

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$4(x - 3) + 2(y + 1) - 5(z - 7) = 0$$

$$4x - 12 + 2y + 2 - 5z + 35 = 0$$

$$4x + 2y - 5z + 25 = 0 \text{ The equation of the plane.}$$

Theorem (10): If a, b, c, and d are constants and a, b, and c are not all zero, then the graph of the equation

$$ax + by + cz + d = 0$$

is a plane having the vector **n**= $\langle a, b, c\rangle$ as a normal.

Equation of a plane through three points:

Example (62): Find the equation of the plane passing through the three points P_1= (1, 2, −1), P_2= (2, 3, 1), and P_3= (3, −1, 2).

Solution:

the equation is, $\quad ax + by + cz + d = 0$

Generally, we can solve this example without using the concepts of the vectors, as following below.

1) the first method:
Since the three points lie in the plane, then their coordinates must satisfy the general equation of the plane, therefore,

$$a + 2b - c + d = 0$$
$$2a + 3b + c + d = 0$$
$$3a - b + 2c + d = 0$$

Now we have homogeneous system of linear equations and we can find the solution of this system using Gauss-Jordan elimination method (**reduced row echelon form**).

<u>note:</u> This homogeneous system of linear equations with more unknowns than equations, therefore, it has infinitely many solutions.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ 3 & -1 & 2 & 1 & 0 \end{bmatrix} -2R_1 + R_2 \to R_2 \text{ and } -3R_1 + R_3 \to R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -7 & 5 & -2 & 0 \end{bmatrix} -1R_2 \to R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & -7 & 5 & -2 & 0 \end{bmatrix} -2R_2 + R_1 \to R_1 \text{ and } 7R_2 + R_3 \to R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & -16 & 5 & 0 \end{bmatrix} -\frac{1}{16}R_3 \to R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5/16 & 0 \end{bmatrix} 3R_3 + R_2 \to R_2 \text{ and } -5R_3 + R_1 \to R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 9/16 & 0 \\ 0 & 1 & 0 & 1/16 & 0 \\ 0 & 0 & 1 & -5/16 & 0 \end{bmatrix}$$

Thus,

$$a + 0 + 0 + \frac{9}{16}d = 0 \to a = -\frac{9}{16}d$$
$$0 + b + 0 + \frac{1}{16}d = 0 \to b = -\frac{1}{16}d$$
$$0 + 0 + c - \frac{5}{16}d = 0 \to c = \frac{5}{16}d$$

Now if $d = -16$, then $a = 9$, $b = 1$, and $c = -5$ (*checking with any equation*).
Therefore, the equation of the plane passing through the three points P_1, P_2, and P_3 is

$$ax + by + cz + d = 0$$

$$9x + y - 5z - 16 = 0 \quad (\textit{checking with any point}).$$

Now we use the concepts of vectors to find the equation of the plane passing through three points.

2) The second method:
3)

Since the points lie in the plane, then the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are parallel to the plane, where

$$\overrightarrow{P_1P_2}= P_2 - P_1= <2 -1, 3-2, 1-(-1)>= <1, 1, 2>;$$

$$\overrightarrow{P_1P_3}= P_3 - P_1= <2, -3, 3>.$$

Therefore, the cross product $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is normal to the plane because it perpendicular to both the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}= \begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \boldsymbol{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix}= <9, 1. -5> \rightarrow a= 9,\ b= 1, \text{ and } c= -5.$$

Since the points P_1, P_2, and P_3 lie in the plane, therefore a **point-normal form for the equation of a plane** is

$$a(x - x_0) + b(y - y_0) + c(z - z_0)= 0$$

We can take any point of these three pointe as point $P_0= (x_0, y_0, z_0)$.

Thus, if we take $P_1= (1, 2, -1)$, we obtain:

$$9(x - 1) + (y - 2) + -5(z + 1)= 0$$

$$9x - 9 + y - 2 - 5z - 5= 0$$

$$9x + y - 5z - 16= 0$$

vector form of equation of a plane:

Vector notation provides a useful alternative way of writing the point-normal form of the equation of a plane.

Let $\mathbf{r}= <x, y, z>$ be the vector from the origin point to the point $P= (x, y, z)$, let $\mathbf{r}_0= <x_0, y_0, z_0>$ the vector from the origin point to the point $P= (x_0, y_0, z_0)$ and let $\mathbf{n}= <a, b, c>$ be a vector normal to the plane, then $\overrightarrow{P_0P}= \mathbf{r} - \mathbf{r}_0$ as shown in (figure 50).

since $\mathbf{n}$ is perpendicular to $\overrightarrow{P_0P}$ then we can write the formula (1) $\mathbf{n} \cdot \overrightarrow{P_0P}= 0$ as:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)= 0 \quad \dots (4)$$

This is called the **vector form of the equation of a plane**.

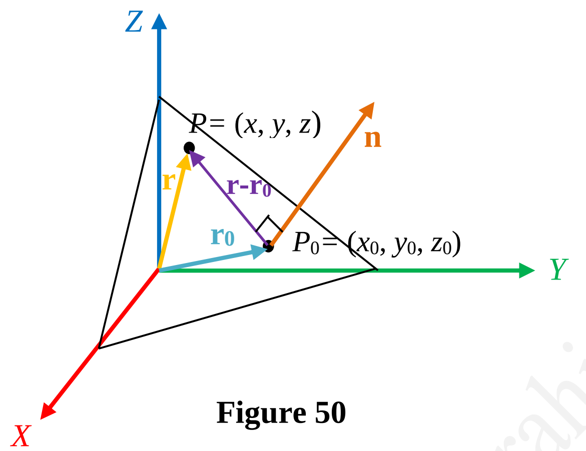

Figure 50

Plane with normal vector

Example (63): Find the vector form of the equation of a plane that passes through the point (6, 3, -4) and perpendicular to the vector **n**= <-1, 2, 5>.

Solution:

$$<-1, 2, 5> . <x - x_0, y - y_0, z - z_0>= 0$$

$$<-1, 2, 5> . <x - 6, y -3, z + 4>= 0$$

$$-1(x - 6) + 2(y -3) + 5(z + 4)= 0$$

$$-x + 6 + 2y - 6 + 5z + 20= 0$$

$$-x + 2y + 5z + 20= 0$$

Example (64): Use the concepts of the vectors to solve the following.

a) If we have the following figure (figure 51), then describe the vector ***AD***.

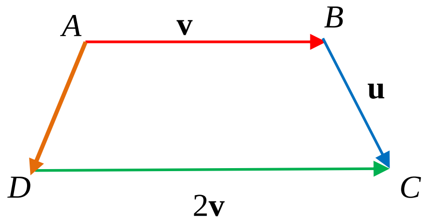

Figure 51

Solution:

$$(\mathbf{v} + \mathbf{u}) - 2\mathbf{v} = -\mathbf{v} + \mathbf{u} = \boldsymbol{AD}.$$

b) *A*, *B*, and *C* are midpoints of their respective lines (vectors), as shown in (figure 52), find the vector ***OB***.

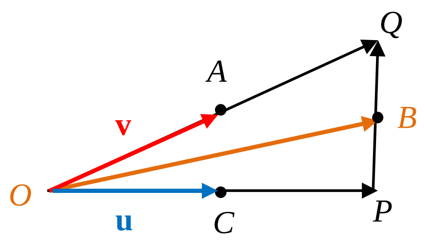

Figure 52

Solution:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \quad \text{or} \quad \overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ} \quad \text{or} \quad \overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$= 2\mathbf{v} - 2\mathbf{u}$$

$$\overrightarrow{PB} = \frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(2\mathbf{v} - 2\mathbf{u}) = \mathbf{v} - \mathbf{u}$$

$$\overrightarrow{OB} = \overrightarrow{OP} + \overrightarrow{PB}$$

$$= 2\mathbf{u} + (\mathbf{v} - \mathbf{u})$$

$$= 2\mathbf{u} + \mathbf{v} - \mathbf{u}$$

$= \mathbf{u} + \mathbf{v}$, as shown in figure 53.

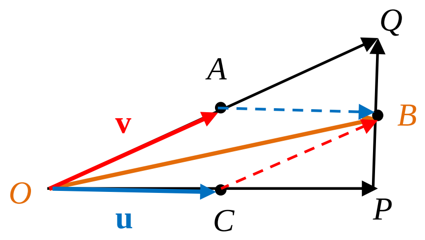

Figure 53

c) If M and N are midpoints of their respective lines (vectors) as shown in figure 54, show that the vectors ***MN*** and ***AB*** are parallel. (***OA***= **v** and ***OB***= **u**).

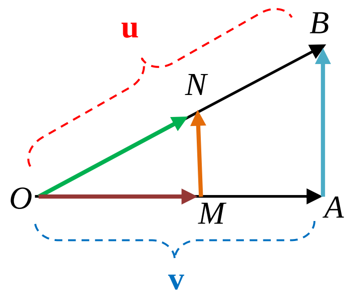

Figure 54

Solution:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\mathbf{v} + \mathbf{u}$$

$$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$$

$$= -\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{u}$$

$$= \frac{1}{2}(-\mathbf{v} + \mathbf{u}) = \frac{1}{2}\overrightarrow{AB}$$

since the vector $\overrightarrow{MN}$ represents the scalar multiple of the vector $\overrightarrow{AB}$, then the vectors are parallel.

References

1- Introductory linear algebra with applications, Bernard Kolman, first edition, 1976.

2- Elementary Linear Algebra Subsequent Edition, Arthur Wayne Roberts,1985.

3- Elementary Linear Algebra, Ninth Edition, Howard Anton, Chris Rorres, 2005.

4- Student Solutions Manuals for use with College Algebra with Trigonometry: graphs and models, by Raymond A. Barnett, Michael R. Ziegler and Karl E. Byleen, 2005.