University of Anbar

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Applied Mathematics

# Numerical Analysis

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Third lecture

4- The Determinant of Matrix

The determinant of square matrix is a number that can be useful in determining the existence and uniqueness of solution to linear systems. We will denote the determinant of a matrix by det A, but it is also common to use the notation |A|.

**Definition** : Suppose that A is a square matrix

1-if A= $[a_1]$ is a $|x|$ matrix, then $|A| = a$

2-if A is an $n \times n$ matrix, with $n > 1$ the minor $M_{ij}$ is determinant of the $(n-1)\times(n-1)$ submatrix of A obtained by determinant of the $ith$ row and $ith$ Column of the matrix A.

3-the cofactor $A_{ij}$ associated with $M_{ij}$ is defined $A_{ij} = (-1)^{i+j} M_{ij}$

4- the determinant of the $n \times n$ matrix A, when $n > 1$ , is given by

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any } i = 1, 2, ..., n$$

Or by

$$|A| = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for any } j = 1, 2, ..., n$$

**Example:** Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 7 & 0 \\ -3 & -4 & 1 & 5 \\ 6 & -6 & 8 & 0 \end{bmatrix}$$

Using the row or column with the most zero entries

**Solution**

To compute $|A|$ , it is easiest to use the fourth column because three of its entries are 0.

$$|A| = a_{14}(-1)^{1+4}M_{14} + a_{24}(-1)^{2+4}M_{24} + a_{34}(-1)^{3+4}M_{34} + a_{44}(-1)^{4+4}M_{44} = -5M_{34}$$

$$|\text{A}| = 0(-1)^{1+4}\begin{vmatrix} 4 & -2 & 7 \\ -3 & -4 & 1 \\ 6 & -6 & 8 \end{vmatrix} + 0(-1)^{2+4}\begin{vmatrix} 2 & -1 & 3 \\ -3 & -4 & 1 \\ 6 & -6 & 8 \end{vmatrix} + 5(-1)^{3+4}\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{vmatrix} + 0(-1)^{4+4}\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ -3 & -4 & 1 \end{vmatrix}$$

$$|A = 0 + 0 - 5\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{vmatrix} + 0 = -5\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{vmatrix}$$

Eliminating the third row and the fourth column of A and expanding the resulting 3x3 matrix by its first row gives

$$|A| = -5\begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & 7 \\ 6 & -6 & 8 \end{vmatrix} = -5\{2(-1)^{1+1} \times \begin{vmatrix} -2 & 7 \\ -6 & 8 \end{vmatrix} (-)^{1+2}(-1)\begin{vmatrix} 4 & 7 \\ 6 & 8 \end{vmatrix} (-)^{1+3}(3)\begin{vmatrix} 4 & -2 \\ 6 & -6 \end{vmatrix}\}$$

$$= -5[2(-16+42) + (32-42) + 3(-24+12)] = -30$$

5- **<u>Matrix Factorization method</u>**

To solve a system of the form $A = xb$ can be to factor a matrix. The factorization is particularly useful when it has the form $A = LU$ . Where $L$ is lower triangular and $U$ is upper triangular .then we can easily solve for x using a two –step process.

1-first define the temporary vector $y = Ux$ and solve the lower triangular system

$$Ly = b \qquad \text{for y.}$$

2-once y is known, the upper triangular system $Ux = y$ to determine the solution x.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = A = LU$$

1- When all element diagonal of the matrix L is one will it called Doolittle and write as follows.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

2- When all element diagonal of the matrix $U$ is one will it called Crout and write as follows.

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The result of multiplying the two matrices (L and U) will be as follows

$$\begin{array}{lll} 1 \times u_{11} = a_{11} & l_{21} \times u_{11} = a_{21} & l_{31} \times u_{11} = a_{31} \\ 1 \times u_{12} = a_{12} & l_{21} \times u_{12} + 1 \times u_{22} = a_{22} & l_{31} \times u_{12} + l_{32} \times u_{22} = a_{32} \\ 1 \times u_{13} = a_{13} & l_{21} \times u_{13} + 1 \times u_{23} = a_{23} & l_{31} \times u_{13} + l_{32} \times u_{23} + 1 \times u_{33} = a_{33} \end{array}$$

after that, we can use the two matrices in two formulas down.

$$Ux = y \qquad \text{(A)}$$

$$Ly = b \qquad \text{(B)}$$

From (B) will obtained y and substitution in (A) to fine the value of x.

**Example :** us crout factorization to solve the linear system .

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 22 \\ 3x_1 + 19x_2 + 17x_3 &= 94 \\ 8x_1 + 36x_2 + 25x_3 &= 166 \end{aligned}$$

**Solution**

$A = LU$

$$\begin{bmatrix} 1 & 5 & 3 \\ 3 & 19 & 17 \\ 8 & 36 & 25 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{lll} l_{11} \times 1 = 1 & l_{21} \times 1 = 3 & l_{31} \times 1 = 8 \\ l_{11} \times u_{12} = 5 \Rightarrow u_{12} = 5 & l_{21} \times u_{12} + l_{22} \times 1 = 19 & l_{31} \times u_{12} + l_{32} = 36 \\ l_{11} \times u_{13} = 3 \Rightarrow u_{13} = 3 & l_{21} \times u_{13} + l_{22} \times u_{23} = 17 & l_{31} \times u_{13} + l_{32} \times u_{23} + l_{33} = 25 \end{array}$$

$$\Rightarrow l_{21} \times u_{12} + l_{22} \times 1 = 19 \Rightarrow 3(5) + l_{22} = 19 \Rightarrow l_{22} = 19 - 15 \Rightarrow l_{22} = 4$$

$$l_{21} \times u_{13} + l_{22} \times u_{23} = 17 \Rightarrow u_{23} = 2$$

$$l_{31} \times u_{12} + l_{32} = 36 \Rightarrow l_{32} = -4$$

$$l_{31} \times u_{13} + l_{32} \times u_{23} + l_{33} = 25 \Rightarrow l_{33} = 9$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 3 & 19 & 17 \\ 8 & 36 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 8 & -4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

No we us

$$Ly = b \qquad \text{(A)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 8 & -4 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 94 \\ 166 \end{bmatrix}$$

$$y_1 = 22$$

$$3y_1 + 4y_2 = 94 \Rightarrow 4y_2 = 94 - 66 = 28 \Rightarrow y_2 = 7$$

$$8y_1 - 4y_2 + 9y_3 = 166 \Rightarrow 9y_3 = 166 + 28 - 176 \Rightarrow y_3 = 2$$

From equation (B) $\Rightarrow Ux = y$ (B)

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \\ 2 \end{bmatrix}$$

$$x_3 = 2$$

$$x_2 + 2x_3 = 7 \Rightarrow x_2 = 3$$

$$x_1 + 5x_2 + 3x_3 = 22 \Rightarrow x_1 = 1$$

Example : use the crout factorization to solve the linear system

a)

$$\begin{aligned} 2x_1 - x_2 + \quad & = 1 \\ -x_1 + 2x_2 - x_3 \quad & = 0 \\ -x_2 + 2x_3 - x_4 & = 0 \\ -x_3 + 2x_4 & = 1 \end{aligned}$$

B)

$$\begin{aligned} 3x_1 + x_2 + \quad & = -1 \\ 2x_1 + 4x_2 + x_3 & = 7 \\ 2x_2 + 5x_3 & = 9 \end{aligned}$$

Example

a) Use the Doolitte factorization for the tridiagonal system to solve the following systems
   B)

$$
\begin{aligned}
2x_1 - x_2 + \quad &= 3 \\
-x_1 + 2x_2 - x_3 &= -3 \\
-x_2 + 2x_3 &= 1
\end{aligned}
$$

$$
\begin{aligned}
4x_1 + x_2 + x_3 + x_4 &= 0.65 \\
x_1 + 3x_2 - x_3 + x_4 &= 0.05 \\
x_1 - x_2 + 2x_3 &= 0 \\
x_1 + x_2 + 2x_4 &= 0.5
\end{aligned}
$$

**Reference**


**1-Numerical Analysis. Richara L. Burden, and J. Douglas Faires .Ninth Edition.**

**2- Numerical Methods. J. Douglas Faires and Richara L. Burden. Fourth Edition.**

**3- Numerical mathematics and Computing. Ward Cheney and David Kincaid. Second Edition.**