University of Anbar

College of Science

Applied Mathematics

# Numerical Analysis

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**Fourth lecture**

**2- Iterative Techniques in Matrix Algebra:**

**2-1- Jacobi's Method.**

The Jacobi iterative method is obtained by solving the ith equation in $Ax = b \;\; for \;\; x_i$ to obtain (provided $a_{ij} \neq 0$ ).

$$x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}}\right) + \frac{b_i}{a_{ii}}, \quad for\, i = 1,2,...,n \qquad (*)$$

For each $k \geq 1$ , generate the components $x_i^{(k)} \; of \;\; x^{(k)}$ **from the components of** $x^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n} (-a_{ij}x_j^{(k-1)} + b_i\right] \;\; for i = 1,2,..,n \qquad (**)$$

The condition for stop

a) $| x_i^{(k)} - x_i^{(k-1)} |_\infty < \in$

b) $\frac{| x_i^{(k)} - x_i^{(k-1)} |_\infty}{| x_i^{(k)} |_\infty} < \in$

**Example use Jacobi's iterative technique to approximation for linear system Ax=b . with** $\in = 10^{-3}$ and initial approximation $x_i^0 = (0,0,0,0) \quad i = 1,2,3,4$

$$\begin{aligned}
E_1 &: 10x_1 - x_2 + 2x_3 &&= 6 \\
E_2 &: -x_1 + 11x_2 - x_3 + 3x_4 &&= 25 \\
E_3 &: 2x_1 - x_2 + 10x_3 - x_4 &&= -11 \\
E_4 &: \quad 3x_2 - x_3 + 8x_4 &&= 15
\end{aligned}$$

**Solution**

$$x_1^{(k)} = \frac{1}{10}(x_2^{(k-1)} - 2x_3^{(k-1)} + 6)$$
$$x_2^{(k)} = \frac{1}{11}(x_1^{(k-1)} + x_3^{(k-1)} - 3x_4^{(k-1)} + 25)$$
$$x_3^{(k)} = \frac{1}{10}(-2x_1^{(k-1)} + x_2^{(k-1)} + x_4^{(k-1)} - 11)$$
$$x_4^{(k)} = \frac{1}{8}(-3x_2^{(k-1)} + x_3^{(k-1)} + 15)$$

For initial approximation and take k=0 starting with

$$x_i^0 = (0,0,0,0) \quad i = 1,2,3,4$$

$$x_1^{(1)} = \frac{1}{10}(x_2^{(1-1)} - 2x_3^{(1-1)} + 6) \qquad = \frac{1}{10}(0 - 2(0) + 6) = \frac{6}{10} = 0.6000$$
$$x_2^{(1)} = \frac{1}{11}(x_1^{(1-1)} + x_3^{(1-1)} - 3x_4^{(1-1)} + 25) = \frac{1}{11}(0 + 0 - 3(0) + 25) = \frac{25}{11} = 2.2727$$
$$x_3^{(1)} = \frac{1}{10}(-2x_1^{(1-1)} + x_2^{(1-1)} + x_4^{(1-1)} - 11) = \frac{1}{10}(-2(0) + 0 + 0 - 11) = -\frac{11}{10} = -1.1000$$
$$x_4^{(1)} = \frac{1}{8}(-3x_2^{(1-1)} + x_3^{(1-1)} + 15) \qquad = \frac{1}{8}(-3(0) + 0 + 15) \qquad = \frac{15}{8} = 1.8750$$

For k=2

$$x_1^{(2)} = \frac{1}{10}(x_2^{(1)} - 2x_3^{(1)} + 6) \qquad = \frac{1}{10}(2.2727 - 2(-1.1000) + 6) = 1.0473$$
$$x_2^{(2)} = \frac{1}{11}(x_1^{(1)} + x_3^{(1)} - 3x_4^{(1)} + 25) = \frac{1}{11}(0.60 - 1.1000 - 3(1.8750) + 25) = 1.7159$$
$$x_3^{(2)} = \frac{1}{10}(-2x_1^{(1)} + x_2^{(1)} + x_4^{(1)} - 11) = \frac{1}{10}(-2(0.60) + 2.2727 + 1.8750 - 11) = -0.8052$$
$$x_4^{(2)} = \frac{1}{8}(-3x_2^{(1)} + x_3^{(1)} + 15) \qquad = \frac{1}{8}(-3(2.2727) - 1.1000 + 15) \qquad = 0.8852$$

For k=3

$$x_1^{(3)} = \frac{1}{10}(x_2^{(2)} - 2x_3^{(2)} + 6) \qquad = \frac{1}{10}(1.7159 - 2(-0.8052) + 6) = 0.9326$$
$$x_2^{(3)} = \frac{1}{11}(x_1^{(2)} + x_3^{(2)} - 3x_4^{(2)} + 25) = \frac{1}{11}(1.0473 - 0.8052 - 3(0.8852) + 25) = 2.053$$
$$x_3^{(3)} = \frac{1}{10}(-2x_1^{(2)} + x_2^{(2)} + x_4^{(2)} - 11) = \frac{1}{10}(-2(1.0473) + 1.7159 + 0.8852 - 11) = -1.0493$$
$$x_4^{(3)} = \frac{1}{8}(-3x_2^{(2)} + x_3^{(2)} + 15) \qquad = \frac{1}{8}(-3(1.7159) - 0.8052 + 15) \qquad = 1.1309$$

For general in a similar manner, we continue until k =10

$$x_1^{10}=1.0001, x_2^{10}=1.9998, x_3^{10}=-0.9998, x_4^{10}=0.9998$$

$$because \quad \frac{|x_4^{10}-x_4^{9}|}{|x_4^{10}|}=\frac{0.0008}{1.9998}=<10^{-3}$$

**Theorem:** the convergence condition (for any iterative method) is when the matrix A is diagonally dominant.

**Definition:** A matrix $A_{m\times n}$ **is said to be diagonally dominant iff , for each i=1,2,…,n**

$$|a_{ii}|>\sum_{j=1, j\neq i}|a_{ij}|$$

**Note to speed convergence the equations should be arranged so that $a_{ii}$ is as large possible**

## 2-2 The Gauss-Seidel Method:

In Jacobi method the components $x^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $x^{(k)}$ . But in Gauss-Seidel method for $i>1$ the component $x_1^k,...,x_{i-1}^k$ *of* $x^{(k)}$ have already bee computed and are expected to be better approximations to the actual solution $x_1,...,x_{i-1}$ *than are* $x_1^{(k-1)},...,x_{i-1}^{(k-1)}$ . So we can use these most recently determined values to compute $x^{(k)}$ that is

$$x_i^{(k)}=\frac{-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)})-\sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)})+b_i}{a_{ii}} \quad for\ i=1,2,..,n$$

**Example: Use the Gauss-Seidel iterative method to fined approximations to**

$$5x_1-2x_2+3x_3=12$$
$$-3x_1+9x_2+x_3=14$$
$$2x_1-x_2-7x_3=-12$$

Choose the initial guess $x^{(0)}=(0,0,0)$

Solution

$$x_1^{(k)}=\frac{1}{5}(12+2x_2^{(k-1)}-3x_3^{(k-1)})$$

$$x_2^{(k)}=\frac{1}{9}(14+3x_1^{(k)}-x_3^{(k-1)})$$

$$x_3^{(k)}=\frac{-1}{7}(-2x_1^{(k)}+x_2^{(k)}-12)$$

For k=1

$$x_1^{(1)} = \frac{1}{5}(12 + 2x_2^{(0)} - 3x_3^{(0)}) = \frac{1}{5}(12 + 2(0) - 3(0)) = 2.4$$

$$x_2^{(1)} = \frac{1}{9}(14 + 3x_1^{(1)} - x_3^{(0)}) = \frac{1}{9}(14 + 3(2.4) - 0) = 2.35555$$

$$x_3^{(1)} = \frac{-1}{7}(-2x_1^{(1)} + x_2^{(1)}) = \frac{-1}{7}(-2(2.4) + (2.35555)) = 2.06349$$

For k=2

$$x_1^{(2)} = \frac{1}{5}(12 + 2x_2^{(1)} - 3x_3^{(1)}) = \frac{1}{5}(12 + 2(2.35555) - 3(2.06349)) = 2.10412$$

$$x_2^{(2)} = \frac{1}{9}(14 + 3x_1^{(2)} - x_3^{(1)}) = \frac{1}{9}(14 + 3(2.10412) - 2.06349) = 2.02765$$

$$x_3^{(2)} = \frac{-1}{7}(-2x_1^{(2)} + x_2^{(2)}) = \frac{-1}{7}(-2(2.10412) + (2.02765)) = 2.02579$$

For k=3

$$x_1^{(3)} = \frac{1}{5}(12 + 2x_2^{(2)} - 3x_3^{(2}) = \frac{1}{5}(12 + 2(2.02765) - 3(2.02579)) = 1.99558$$

$$x_2^{(3)} = \frac{1}{9}(14 + 3x_1^{(3)} - x_3^{(2)}) = \frac{1}{9}(14 + 3(1.99558) - 2.02579) = 1.99566$$

$$x_3^{(3)} = \frac{-1}{7}(-2x_1^{(3)} + x_2^{(3)}) = \frac{-1}{7}(-2(1.99558) + (1.99566)) = 1.99935$$

**Reference**

**1-Numerical Analysis. Richara L. Burden, and J. Douglas Faires .Ninth Edition.**

**2- Numerical Methods. J. Douglas Faires and Richara L. Burden. Fourth Edition.**

**3- Numerical mathematics and Computing. Ward Cheney and David Kincaid. Second Edition.**