

التفاضل والتكامل 2 (الفصل الثاني) – المستوى الاول – طلبة قسم الرياضيات التطبيقية - كلية العلوم – جامعة الانبار
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جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل 2

للطلبة المرحلة الاولى

الفصل الخامس: المحاضرة الثانية

(مجموع ريمان)

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التفاضل والتكامل 2 . د. ملاذ رحيم

CH. 5: Integration

5.2 Sigma Notation and Limits of Finite Sums

Def. The sigma notation for the finite sum $a_1 + a_2 + \dots + a_n$ is denoted by $\sum_{k=1}^n a_k$, a_1 is the first term, a_2 is the second term, ..., a_k the k th term.

Ex.(1) Evaluate the value of the following summations:

$$a) \sum_{k=1}^5 k^2 \quad b) \sum_{k=2}^6 (k-1) \quad c) \sum_{k=1}^4 (-1)^k \cos(k\pi)$$

$$\text{Sol. } a) \sum_{k=1}^5 k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$b) \sum_{k=2}^6 (k-1) = (2-1) + (3-1) + (4-1) + (5-1) + (6-1) \\ = 1 + 2 + 3 + 4 + 5 = 15$$

$$c) \sum_{k=1}^4 (-1)^k \cos(k\pi) = (-1)^1 \cos \pi + (-1)^2 \cos 2\pi + \\ (-1)^3 \cos 3\pi + (-1)^4 \cos 4\pi \\ = (-1)(-1) + 1 + (-1)(-1) + 1 = 4$$

Ex. (2) Express the following sums in sigma notations:

$$a) 1 + 3 + 5 + 7 + 9 + 11 + 13 = \sum_{k=1}^7 (2k - 1)$$

$$b) -\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} + \frac{6}{5} = \sum_{k=1}^6 (-1)^k \frac{k}{5}$$

Algebra Rules for Finite Sums

1. $\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k$
2. $\sum_{k=1}^n c a_k = c \cdot \sum_{k=1}^n a_k$
3. $\sum_{k=1}^n c = n \cdot c$ (where c is constant)

Ex.(3) Find the value of $\sum_{k=1}^7 \pi$.

$$\text{Sol. } \sum_{k=1}^7 \pi = 7\pi \quad (\text{rule 3})$$

Ex.(4) Suppose that $\sum_{k=1}^n a_k = -5$ and $\sum_{k=1}^n b_k = 6$, find the value of $\sum_{k=1}^n (b_k - 2a_k)$

$$\text{Sol. } \sum_{k=1}^n (b_k - 2a_k) = \sum_{k=1}^n b_k - \sum_{k=1}^n 2a_k \quad (\text{rule 1}) \\ = \sum_{k=1}^n b_k - 2 \sum_{k=1}^n a_k \quad (\text{rule 2})$$

$$= 6 - 2(-5) = 6 + 10 = 16$$

Some Formulas for Positive Integers

1. $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
2. $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Ex.(5) Evaluate the following sums:

a) $\sum_{k=1}^7 k(2k+1)$ b) $(\sum_{k=1}^7 k)^2 - \sum_{k=1}^7 \frac{k^3}{4}$

Sol. a) $\sum_{k=1}^7 k(2k+1) = \sum_{k=1}^7 (2k^2 + k) = 2 \sum_{k=1}^7 k^2 + \sum_{k=1}^7 k$

$$= 2 \left[\frac{7(7+1)(14+1)}{6} \right] + \frac{7(7+1)}{2} = \frac{7 \cdot 8 \cdot 15}{3} + \frac{7 \cdot 8}{2}$$

$$= 7 \cdot 8 \cdot 5 + 7 \cdot 4 = 280 + 28 = 308$$

b) $(\sum_{k=1}^7 k)^2 - \sum_{k=1}^7 \frac{k^3}{4} = \left(\frac{7(7+1)}{2}\right)^2 - \frac{1}{4} \sum_{k=1}^7 k^3$

$$= \frac{49 \cdot 64}{4} - \frac{1}{4} \frac{7^2(7+1)^2}{4} = 49 \cdot 16 - \frac{49 \cdot 64}{16}$$

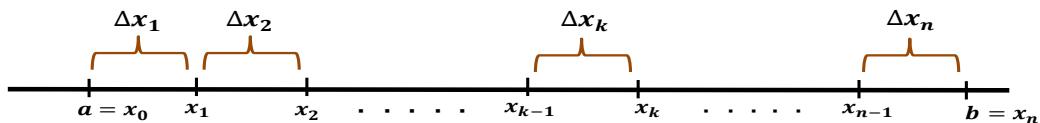
$$= 49 \cdot 16 - 49 \cdot 4 = 49(16 - 4)$$

$$= 49 \cdot 12 = 588$$

Def. Let x_1, x_2, \dots, x_{n-1} be points between a and b such that

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

- 1) the set $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ is called a partition of $[a, b]$
- 2) the length of the k th subinterval $[x_{k-1}, x_k]$ is $\Delta x = x_k - x_{k-1}$.



Ex.(6) Consider the interval $[2, 7]$, let $P = \{2, 3, 4.5, 5, 7\}$ then

$$\Delta x_1 = x_1 - x_0 = 3 - 2 = 1, \Delta x_2 = 4.5 - 3 = 1.5$$

$$\Delta x_3 = 5 - 4.5 = 0.5, \Delta x_4 = 7 - 5 = 2$$

Def. (Riemann Sums)

Let f be a continuous function on $[a, b]$, let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$ and let $c_k \in [x_{k-1}, x_k]$, then a *Riemann Sum* for the area enclosed by the function f on $[a, b]$ is defined as

$$S_p = \sum_{k=1}^n f(c_k) \Delta x_k = f(c_1) \Delta x_1 + f(c_2) \Delta x_2 + \dots + f(c_n) \Delta x_n$$

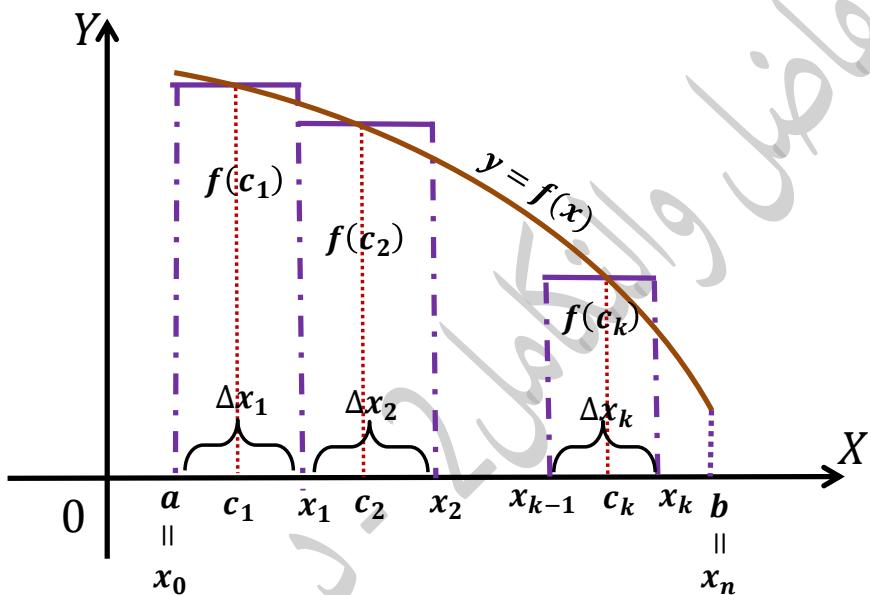


Figure 5.5 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x -axis.

Def. The norm of partition $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ is

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n\}.$$

Ex.(7) Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$

$$\text{Sol. } \Delta x_1 = -1.6 - (-2) = 0.4, \Delta x_2 = -0.5 - (-1.6) = 1.1,$$

$$\Delta x_3 = 0 - (-0.5) = 0.5, \Delta x_4 = 0.8 - 0 = 0.8, \Delta x_5 = 1 - 0.8 = 0.2$$

$$\text{So, } \|P\| = \max\{0.4, 1.1, 0.5, 0.8, 0.2\} = 1.1$$

Ex.(8) Let $f(x) = x^2 + 1$ over the interval $[0, 3]$

- a) Find a formula for **Riemann sum** by dividing the interval $[0, 3]$ into n equal subintervals and using the right hand endpoint for each c_k .
- b) Take a limit of the sum as $n \rightarrow \infty$ to calculate the area under the curve on $[0, 3]$.

Sol.a) The width each of subinterval is $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ (*Figure 5.6*)

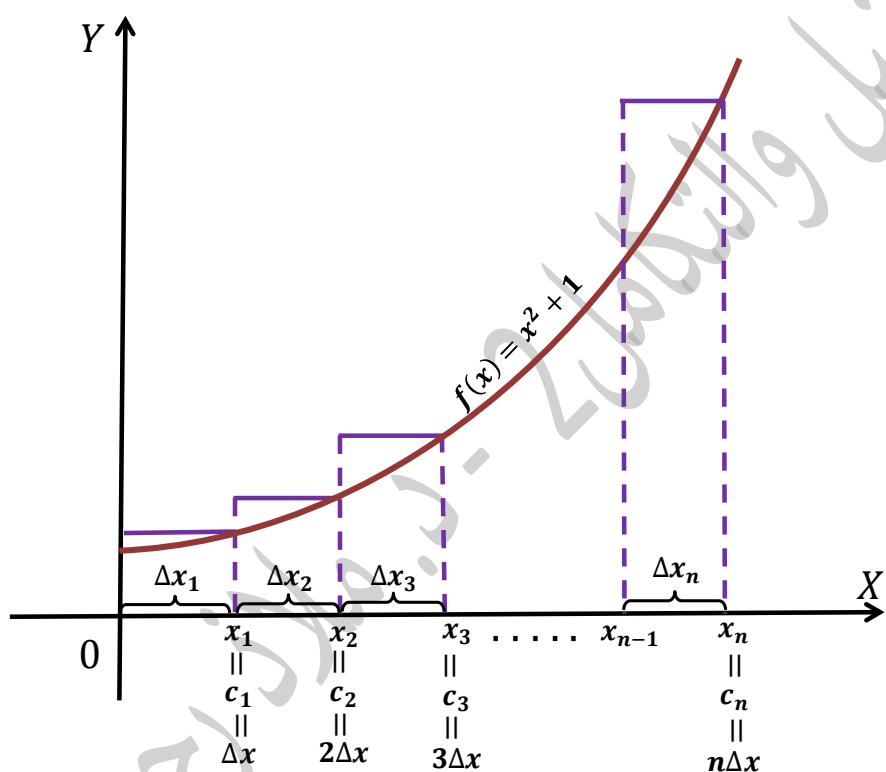


Figure 5.6 Riemann sum for f by dividing the interval $[0, 3]$ into n equal subintervals and using the right hand endpoint for each c_k .

$$\begin{aligned}
 S_p &= \sum_{k=1}^n f(c_k)\Delta x_k = f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x \\
 &= \Delta x [f(\Delta x) + f(2\Delta x) + f(3\Delta x) + \cdots + f(n\Delta x)] \\
 &= \frac{3}{n} \left[f\left(\frac{3}{n}\right) + f\left(2\left(\frac{3}{n}\right)\right) + f\left(3\left(\frac{3}{n}\right)\right) + \cdots + f\left(n\left(\frac{3}{n}\right)\right) \right]
 \end{aligned}$$

$$\begin{aligned}&= \frac{3}{n} \left[\left(\frac{3}{n} \right)^2 + 1 + 2^2 \left(\frac{3}{n} \right)^2 + 1 + 3^2 \left(\frac{3}{n} \right)^2 + 1 + \dots + n^2 \left(\frac{3}{n} \right)^2 + 1 \right] \\&= \frac{3}{n} \left[\left(\frac{3}{n} \right)^2 (1^2 + 2^2 + 3^2 + \dots + n^2) + (1 + 1 + 1 + \dots + 1) \right] \\&= \frac{27}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{3}{n} (n) = \frac{9}{2n^2} (n+1)(2n+1) + 3 \\&= \frac{9}{2n^2} (2n^2 + 3n + 1) + 3 = 9 \left(1 + \frac{3n}{2n^2} + \frac{1}{2n^2} \right) + 3 \\&= 9 + \frac{27n+9}{2n^2} + 3 = 12 + \frac{27n+9}{2n^2}\end{aligned}$$

b) Area = $\lim_{n \rightarrow \infty} S_p = \lim_{n \rightarrow \infty} \left(12 + \frac{27n+9}{2n^2} \right) = 12 + 0 = 12$

Exercises 5.2:

1. Write the sums in the following without sigma notation. Then evaluate them. (*Hint: Ex. 1*)

$$a) \sum_{k=1}^2 \frac{6k}{k+1} \quad b) \sum_{k=1}^4 \cos k\pi \quad c) \sum_{k=1}^3 (-1)^{k+1} \sin \frac{\pi}{k}$$

Answer: a) 7 , b) 0 , c) $\frac{\sqrt{3}-2}{2}$

2. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation? a) $\sum_{k=1}^6 2^{k-1}$ b) $\sum_{k=0}^5 2^k$ c) $\sum_{k=-1}^4 2^{k+1}$

3. Which formula is not equivalent to the other two?

$$a) \sum_{k=2}^4 \frac{(-1)^{k-1}}{k-1} \quad b) \sum_{k=0}^2 \frac{(-1)^k}{k+1} \quad c) \sum_{k=-1}^1 \frac{(-1)^k}{k+2}$$

4. Express the following sums in sigma notation. (*Hint: Ex. 2*)

$$a) \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad b) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

5. Suppose that $\sum_{k=1}^n a_k = 0$ and $\sum_{k=1}^n b_k = 1$. Find the values of:

$$a) \sum_{k=1}^n 8a_k , \quad b) \sum_{k=1}^n 250b_k , \quad c) \sum_{k=1}^n (a_k + 1) , \quad d) \sum_{k=1}^n (b_k - 1)$$

(*Hint: Ex. 4*) Answer: a) 0 , b) 250 , c) n , d) 1 - n

6. Evaluate the sums in the following. (*Hint: Ex. 5*)

$$a) \sum_{k=1}^{13} k^2 , \quad b) \sum_{k=1}^{13} k^3 , \quad c) \sum_{k=1}^6 (3 - k^2) , \quad d) \sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3$$

Answer: a) 819 , b) 8281 , c) - 73 , d) 3376

7. For the function in the following, find a formula (*Riemann sum*) for the upper sum obtained by dividing the interval $[a, b]$ into n equal subintervals. Then take a limit of these sums as $n \rightarrow \infty$ to calculate the area under the curve over $[a, b]$. (*Hint: Ex. 8*)

a) $f(x) = 2x$ over the interval $[0, 3]$.

b) $f(x) = 3x^2$ over the interval $[0, 1]$.

c) $f(x) = x + x^2$ over the interval $[0, 1]$.

Answer: a) 9 , b) 1 , c) $\frac{5}{6}$

المصادر:

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