

جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل 2

للطلبة المرحلة الاولى

الفصل الخامس: المحاضرة الثالثة

(التكامل المحدد)

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CH. 5: Integration

5.3 The Definite Integral

Def. Let $f(x)$ be a function defined on a closed interval $[a, b]$. If $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ is a partition of $[a, b]$ we define the integral of f over $[a, b]$ by $\int_a^b f(x)dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x_k$, provided the limit exists.

Note: If we choose a partition with n equal subintervals, then the width of each subinterval is $\Delta x = \frac{b-a}{n}$ and so the above define become:

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x_k = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\left(\frac{b-a}{n}\right) \\ &= (b-a) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} f(c_k).\end{aligned}$$

Ex (1) Use Riemann sum to evaluate the definite integral $\int_0^2 (2x + 1)dx$.

Sol. $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

Take c_k at the right end of each subinterval (*Figure 5.7*)

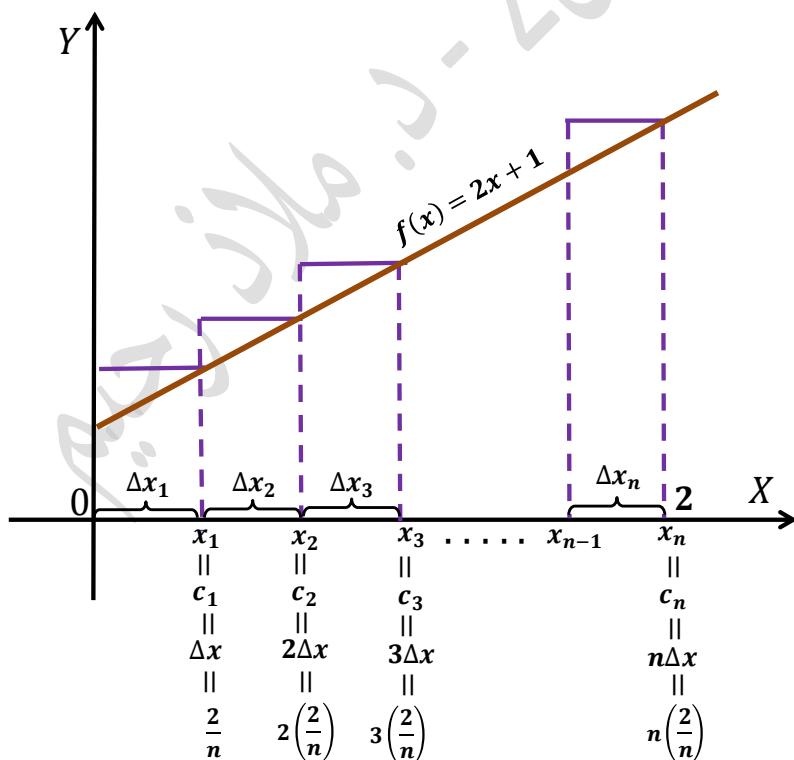


Figure 5.7 Riemann sum for f by dividing the interval $[0, 2]$ into n equal subintervals and using the right hand endpoint for each c_k .

$$\begin{aligned}
 S_P &= f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x \\
 &= \Delta x[f(\Delta x) + f(2\Delta x) + f(3\Delta x) + \cdots + f(n\Delta x)] \\
 &= \frac{2}{n} \left[f\left(\frac{2}{n}\right) + f\left(2\left(\frac{2}{n}\right)\right) + f\left(3\left(\frac{2}{n}\right)\right) + \cdots + f\left(n\left(\frac{2}{n}\right)\right) \right] \\
 &= \frac{2}{n} \left[2\left(\frac{2}{n}\right) + 1 + 2\left(2\left(\frac{2}{n}\right)\right) + 1 + 2\left(3\left(\frac{2}{n}\right)\right) + 1 + \cdots + 2\left(n\left(\frac{2}{n}\right)\right) + 1 \right] \\
 &= \frac{2}{n} \left[\frac{4}{n} + 2\left(\frac{4}{n}\right) + 3\left(\frac{4}{n}\right) + \cdots + n\left(\frac{4}{n}\right) + (1 + 1 + 1 + \cdots + 1) \right] \\
 &= \frac{2}{n} \left(\frac{4}{n} \right) (1 + 2 + 3 + \cdots + n) + \frac{2}{n} (n) = \frac{8}{n^2} \left(\frac{n(n+1)}{2} \right) + 2 \\
 &= \frac{4}{n} (n + 1) + 2 = 4 + \frac{4}{n} + 2 = 6 + \frac{4}{n} \\
 \int_0^2 (2x + 1) dx &= \lim_{n \rightarrow \infty} \left(6 + \frac{4}{n} \right) = 6
 \end{aligned}$$

Ex.(2) Express $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 - 3c_k)\Delta x_k$, where P is a partition of the interval $[-7, 5]$ as definite integral.

Sol. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 - 3c_k)\Delta x_k = \int_{-7}^5 f(x) dx = \int_{-7}^5 (x^2 - 3x) dx$

Theorem(5.1): If f is continuous function on $[a, b]$, then $\int_a^b f(x) dx$ exist that is f integrable on $[a, b]$.

Properties of definite integrals

If f and g are integrable functions over $[a, b]$, then

1. $\int_b^a f(x) dx = - \int_a^b f(x) dx$
2. $\int_a^a f(x) dx = 0$
3. $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
4. $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

5. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

6. If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$,

$$\text{then } \min f \cdot (b - a) \leq \int_a^b f(x)dx \leq \max f \cdot (b - a)$$

7. If $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

If $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)dx \geq 0$

Proof of Rule 6: Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never large than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k ,

$$\begin{aligned} \min f \cdot (b - a) &= \min f \cdot \sum_{k=1}^n \Delta x_k \\ &= \sum_{k=1}^n \min f \cdot \Delta x_k \\ &\leq \sum_{k=1}^n f(c_k) \Delta x_k \\ &\leq \sum_{k=1}^n \max f \cdot \Delta x_k \\ &= \max f \cdot \sum_{k=1}^n \Delta x_k \\ &= \max f \cdot (b - a). \end{aligned}$$

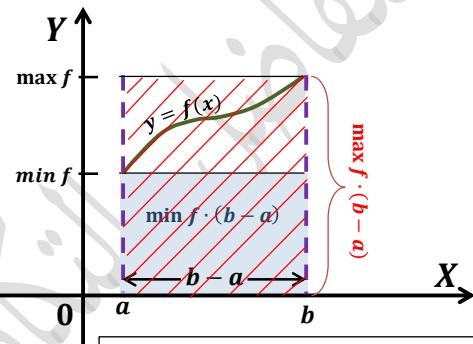


Figure 5.8 Max-Min Inequality

$$\sum_{k=1}^n \Delta x_k = b - a$$

Constant Multiple Rule

$$\min f \leq f(c_k)$$

$$f(c_k) \leq \max f$$

Constant Multiple Rule

Ex.(3) If f and g are integrable functions and $\int_1^2 f(x)dx = -4$,

$\int_1^5 f(x)dx = 6$ and $\int_1^5 g(x)dx = 8$ find the values of:

a) $\int_2^1 -2f(x)dx$ b) $\int_2^5 5f(x)dx$ c) $\int_1^5 (3f(x) - g(x))dx$

Sol. a) $\int_2^1 -2f(x)dx = -\int_1^2 -2f(x)dx = \int_1^2 2f(x)dx$

$$= 2 \int_1^2 f(x)dx = 2(-4) = -8$$

b) $\int_2^5 5f(x)dx = 5 \int_2^5 f(x)dx$

$$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^5 f(x)dx \rightarrow 6 = -4 + \int_2^5 f(x)dx$$

$$\rightarrow \int_2^5 f(x)dx = 10 \rightarrow \int_2^5 5f(x)dx = 5 \cdot 10 = 50$$

c) $\int_1^5 (3f(x) - g(x))dx = 3 \int_1^5 f(x)dx - \int_1^5 g(x)dx = 3(6) - 8 = 10$

Area Under the Graph of a Nonnegative Function

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ and the X -axis over $[a, b]$ is the integral of f from a to b ,

$$A = \int_a^b f(x)dx$$

Ex.(4) Graph the integrand and use area to evaluate the following integrals

a) $\int_{-4}^0 \sqrt{16 - x^2} dx$ b) $\int_{-1}^1 (2 - |x|)dx$ c) $\int_{-1}^1 (1 + \sqrt{1 - x^2})dx$

Sol.a) (Figure 5.9) $\int_{-4}^0 \sqrt{16 - x^2} dx = \frac{1}{4}$ area of the circle $\frac{1}{4}(\pi 4^2) = 4\pi$

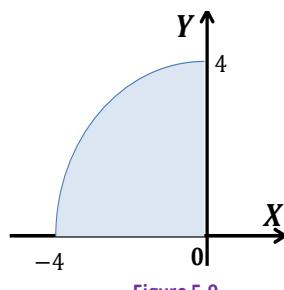


Figure 5.9

Some Basic Integrals

$$1. \int_0^b x \, dx = \frac{b^2}{2}$$

$$2. \int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad a < b$$

$$3. \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b$$

$$4. \int_a^b c \, dx = (b - a) c, \quad c \text{ is constant}$$

Ex.(5) Evaluate the following integrals:

$$a) \int_2^5 x^2 \, dx \qquad b) \int_0^1 (2x - x^3) \, dx \qquad c) \int_1^2 \left(\frac{x}{2} + 2\right) \, dx$$

$$\underline{\text{Sol.}} \quad a) \int_2^5 x^2 \, dx = \frac{5^3}{3} - \frac{2^3}{3} = \frac{1}{3}(125 - 8) = \frac{117}{3} = 39$$

$$\begin{aligned} b) \int_0^1 (2x - x^3) \, dx &= 2 \int_0^1 x \, dx - \int_0^1 x^3 \, dx \\ &= 2 \left(\frac{1^2}{2}\right) - \frac{1^4}{4} = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} c) \int_1^2 \left(\frac{x}{2} + 2\right) \, dx &= \frac{1}{2} \int_1^2 x \, dx + \int_1^2 2 \, dx \\ &= \frac{1}{2} \left(\frac{2^2}{2} - \frac{1^2}{2}\right) + 2(2 - 1) = \frac{1}{2} \left(2 - \frac{1}{2}\right) + 2 \\ &= \frac{1}{2} \left(\frac{3}{2}\right) + 2 = \frac{3}{4} + 2 = \frac{11}{4} \end{aligned}$$

Def. If f is integrable on $[a, b]$, then its average value on $[a, b]$ is defined as

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Ex.(6) Find the average value of the function $f(x) = (x - 1)^2$ over $[0,3]$

$$\begin{aligned}\textbf{Sol. } \text{av}(f) &= \frac{1}{3-0} \int_0^3 (x-1)^2 dx = \frac{1}{3} \int_0^3 (x^2 - 2x + 1) dx \\ &= \frac{1}{3} \left[\int_0^3 x^2 dx - 2 \int_0^3 x dx + \int_0^3 1 dx \right] \\ &= \frac{1}{3} \left[\frac{3^3}{3} - 2 \left(\frac{3^2}{2} \right) + 1(3 - 0) \right] \\ &= \frac{1}{3} [9 - 9 + 3] = \frac{1}{3} \cdot 3 = 1\end{aligned}$$

المصادر:

- التفاضل والتكامل، جورج ثوماس، 12، بيرسون-دلهي، 2009
- سلسلة شوم-حساب التفاضل والتكامل، اليوت مندلسون، اكاديميا انترناشونال، 2006
- حسان التفاضل والتكامل، باسل الهاشمي

التفاضل والتكامل 2 . د. ملاذ رحيم