

جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل 2

للطلبة المرحلة الاولى

الفصل الخامس: المحاضرة الرابعة

(النظرية الاساسية في حساب التفاضل والتكامل)

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## CH. 5: Integration

### 5.4 The Fundamental Theorem of Calculus

#### **Theorem(5.2): (The Mean Value Theorem for Definite Integrals)**

If  $f$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$ ,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

**Ex.(1)** Let  $f(x) = x^2$ ,  $x \in [0, 9]$

- 1) Find  $av(f)$  on the given interval.
- 2) Find a point  $c$  in  $[0, 9]$  at which the given function take this average value.

**Sol.**

$$1) av(f) = \frac{1}{9-0} \int_0^9 x^2 dx = \frac{1}{9} \left[ \frac{x^3}{3} \right] = \frac{9^2}{3} = \frac{81}{3} = 27$$

$$2) f(c) = \frac{1}{b-a} \int_a^b f(x) dx \rightarrow c^2 = av(f) \rightarrow c^2 = 27 \rightarrow c = \pm 3\sqrt{3}$$

$$-3\sqrt{3} \notin [0, 9], \text{ so } c = 3\sqrt{3} \in [0, 9]$$

#### **Theorem(5.3): ( Fundamental Theorem of Calculus, part 1)**

If  $f$  is continuous on  $[a, b]$  then  $F(x) = \int_a^x f(t) dt$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and its derivative is  $f(x)$ :

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

#### **Corollary:**

$$1) \frac{d}{dx} \left( \int_a^{g(x)} f(t) dt \right) = f(g(x)) \cdot g'(x)$$

$$2) \frac{d}{dx} \left( \int_{h(x)}^{g(x)} f(t) dt \right) = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

**Ex.(2) Find the derivatives of the following functions**

a)  $y = \int_x^1 t \cos t dt$

Sol.

$$y = - \int_1^x t \cos t dt \Rightarrow y' = -x \cos x$$

b)  $y = \int_{\sqrt{x}}^0 \sin(t^2) dt$

Sol.

$$y = - \int_0^{\sqrt{x}} \sin(t^2) dt \Rightarrow y' = - \sin((\sqrt{x})^2) \cdot \frac{d}{dx}(\sqrt{x}) = \frac{-\sin x}{2\sqrt{x}}$$

$$2x^3 \cdot \cos(x^6) + 2x \int_2^{x^2} \cos(t^3) dt$$

**Ex.(3) If**

$$y = \sin x \int_{\cot x}^{\tan x} \frac{1}{1+t^2} dt, \text{ find } y' \text{ at } x = \frac{\pi}{4}$$

Sol.

$$\begin{aligned} y' &= \sin x \left[ \frac{1}{1+\tan^2 x} \cdot \sec^2 x - \frac{1}{1+\cot^2 x} \cdot (-\csc^2 x) \right] \\ &\quad + \cos x \int_{\cot x}^{\tan x} \frac{1}{1+t^2} dt \end{aligned}$$

$$= \sin x [1+1] + \cos x \int_{\cot x}^{\tan x} \frac{1}{1+t^2} dt = 2 \sin x + \cos x \int_{\cot x}^{\tan x} \frac{1}{1+t^2} dt$$

$$y' \left( \frac{\pi}{4} \right) = 2 \left( \frac{1}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \int_1^1 \frac{1}{1+t^2} dt = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

### ( The Evaluation Theorem)

**Theorem(5.4): ( Fundamental Theorem of Calculus, part 2)**

If  $f$  is continuous at every point in  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$  then

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

**Ex.(4) Calculate the following integrals.**

$$1) \int_0^1 (x^2 + \sqrt{x}) dx = \int_0^1 \left( x^2 + x^{\frac{1}{2}} \right) dx = \left[ \frac{x^3}{3} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \left[ \frac{x^3}{3} + \frac{2}{3} x^{\frac{3}{2}} \right]_0^1$$

$$= \frac{(1)^3}{3} + \frac{2}{3}(1)^{\frac{3}{2}} - 0 - 0 = \frac{1}{3} + \frac{2}{3} = \boxed{1}$$

$$2) \int_0^\pi (1 + \cos x) dx = [x + \sin x]_0^\pi = \pi + \sin \pi - 0 - \sin 0 = \boxed{\pi}$$

$$3) \int_1^2 \left( x + \frac{1}{x} \right)^2 dx = \int_1^2 \left( x^2 + 2 + \frac{1}{x^2} \right) dx = \int_1^2 (x^2 + 2 + x^{-2}) dx$$

$$= \left[ \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} \right]_1^2 = \left[ \frac{x^3}{3} + 2x - \frac{1}{x} \right]_1^2 = \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} - 2 + 1$$

$$= 3 + \frac{7}{3} - \frac{1}{2} = \frac{18 + 14 - 3}{6} = \boxed{\frac{29}{6}}$$

## Total Area

Remark: To find the total area between the graph of the function  $y = f(x)$  and the  $X$ -axis over the interval  $[a, b]$ , we make the following steps:

- 1) Subdivide the interval  $[a, b]$  at the zeros of  $f$ .
- 2) Integrate  $f$  over each subinterval.
- 3) Add the absolute values of the integrals

$$\text{Total area } A = |A_1| + |A_2| + |A_3|$$

$$A_1 = \int_a^{x_1} f(x)dx, \quad A_2 = \int_{x_1}^{x_2} f(x)dx, \quad A_3 = \int_{x_2}^b f(x)dx$$

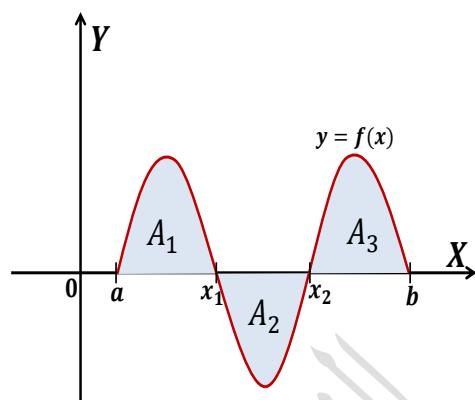


Figure 5.12 The total area between  $y=f(x)$  and the  $X$ -axis for  $a \leq x \leq b$

**Ex.(5) Find the total area between the  $X$ -axis and the graph of the function  $f(x) = x^3 - 3x^2 + 2x$  on the interval  $[-2, 2]$ .**

**Sol.** Figure 5.13

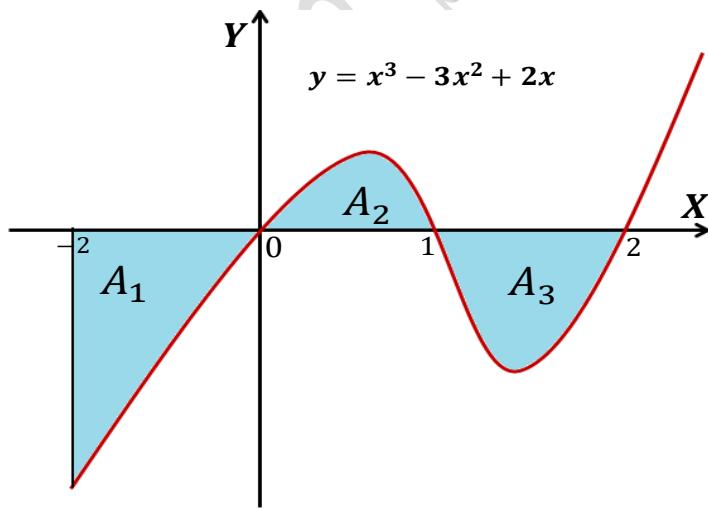
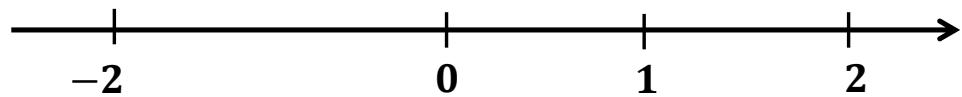


Figure 5.13 The total area between  $f(x) = x^3 - 3x^2 + 2x$  and the  $X$ -axis for  $-2 \leq x \leq 2$

$$f(x) = x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0$$

$$x(x - 1)(x - 2) = 0 \Rightarrow x = 0, x = 1, x = 2$$



$$\begin{aligned}
 A_1 &= \int_{-2}^0 (x^3 - 3x^2 + 2x) dx = \left[ \frac{x^4}{4} - x^3 + x^2 \right]_{-2}^0 \\
 &= 0 - 0 + 0 - \left[ \frac{16}{4} + 8 + 4 \right] = \boxed{-16} \\
 A_2 &= \int_0^1 (x^3 - 3x^2 + 2x) dx = \left[ \frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{1}{4} - 1 + 1 - 0 = \boxed{\frac{1}{4}} \\
 A_3 &= \int_1^2 (x^3 - 3x^2 + 2x) dx = \left[ \frac{x^4}{4} - x^3 + x^2 \right]_1^2 \\
 &= \frac{16}{4} - 8 + 4 - \left[ \frac{1}{4} - 1 + 1 \right] = 4 - 8 + 4 - \frac{1}{4} + 1 - 1 = \boxed{-\frac{1}{4}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Total area} &= |A_1| + |A_2| + |A_3| = |-16| + \left| \frac{1}{4} \right| + \left| -\frac{1}{4} \right| \\
 &= 16 + \frac{1}{4} + \frac{1}{4} = 16 + \frac{1}{2} = \boxed{\frac{33}{2}}
 \end{aligned}$$

Exercises 5.4:

1. Evaluate the integrals in the following

$$a) \int_{-2}^0 (2x + 5) dx$$

$$b) \int_0^4 \left(3x - \frac{x^3}{4}\right) dx$$

$$c) \int_0^1 (x^2 + \sqrt{x}) dx$$

$$d) \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \csc^2 x dx$$

$$e) \int_0^{\frac{\pi}{3}} 4 \sec u \tan u du$$

$$f) \int_{\frac{\pi}{2}}^0 \frac{1 + \cos 2t}{2} dt$$

$$g) \int_{-\frac{\pi}{3}}^{-\frac{\pi}{4}} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$$

$$h) \int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$$

$$i) \int_{-4}^4 |x| dx$$

Answer: a) 6, b) 8, c) 1, d)  $2\sqrt{3}$ , e) 4, f)  $-\frac{\pi}{4}$ , g)  $4\sqrt{3} - 3$ ,

$$h) \sqrt{2} - \sqrt[4]{8} + 1, i) 16$$

2. Find the derivatives: i) by evaluating the integral and differentiating the result. ii) by differentiating the integral directly.

$$a) \frac{d}{dx} \int_1^{\sin x} 3t^2 dt$$

$$\text{Sol. i)} \int_1^{\sin x} 3t^2 dt = [t^3]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx} \left( \int_1^{\sin x} 3t^2 dt \right) = \frac{d}{dx} (\sin^3 x - 1) = 3 \sin^2 x \cos x$$

$$\text{ii)} \frac{d}{dx} \left( \int_1^{\sin x} 3t^2 dt \right) = 3 \sin^2 x \cos x$$

$$b) \frac{d}{dx} \int_0^{\sqrt{x}} \cos t dt$$

$$c) \frac{d}{dt} \int_0^{t^2} \sqrt{u} du$$

3. Find  $\frac{dy}{dx}$  in the following: (Hint: Ex. 2)

$$a) y = \int_0^x \sqrt{1+t^2} dt \quad b) y = \int_{\sqrt{x}}^0 \sin(t^2) dt \quad c) y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, |x| < \frac{\pi}{2}$$

المصادر:

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