

جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل 2

للطلبة المرحلة الاولى

الفصل الخامس: المحاضرة السادسة

(طريقة التعويض والتكامل المحدد)

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CH. 5: Integration

5.6 Substitution and Definite Integrals

Theorem(5.5): (Substitution in Definite Integrals)

If g' is continuous on $[a, b]$, and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

$u = g(x) \rightarrow du = g'(x)dx$

Ex.(2) Evaluate the following integrals.

a) $\int_{-1}^1 x^3 (1 + x^4)^3 dx$

Sol. $t = 1 + x^4 \rightarrow dt = 4x^3 dx \rightarrow \frac{1}{4}dt = x^3 dx$

$$= \frac{1}{4} \int_2^2 t^3 dt = 0$$

b) $\int_{-1}^0 \frac{x^3}{\sqrt{x^4 + 9}} dx$

Sol. $w = x^4 + 9 \rightarrow dw = 4x^3 dx \rightarrow \frac{1}{4}dw = x^3 dx$

$$= \frac{1}{4} \int_{10}^9 \frac{dw}{\sqrt{w}} = -\frac{1}{4} \int_9^{10} w^{-\frac{1}{2}} dw = -\frac{1}{4} \left[\frac{w^{\frac{1}{2}}}{\frac{1}{2}} \right]_9^{10} = -\frac{1}{2} [\sqrt{10} - 3]$$

c) $\int_0^1 \frac{10\sqrt{x}}{\left(1+x^{\frac{3}{2}}\right)^2} dx$

Sol. $y = 1 + x^{\frac{3}{2}} \rightarrow dy = \frac{3}{2}x^{\frac{1}{2}}dx \rightarrow \frac{2}{3}dy = \sqrt{x}dx \rightarrow \frac{20}{3}dy = 10\sqrt{x}dx$

$$= \frac{20}{3} \int_1^2 \frac{dy}{y^2} = \frac{20}{3} \left[-\frac{1}{y} \right]_1^2 = -\frac{20}{3} \left[\frac{1}{2} - 1 \right] = -\frac{20}{3} \left(-\frac{1}{2} \right) = \frac{10}{3}$$

Theorem(5.6): (Definite Integrals of Symmetric Functions)

Let f be continuous on the symmetric interval $[-a, a]$.

a) If f is even, then $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$.

b) If f is odd, then $\int_{-a}^a f(x)dx = 0$

Ex.(3) Find the value of the following integrals.

a) $\int_{-2}^2 \frac{3x}{(9+x^2)^2} dx$

Sol. $f(x) = \frac{3x}{(9+x^2)^2} \rightarrow f(-x) = \frac{3(-x)}{(9+(-x)^2)^2} = -\left[\frac{3x}{(9+x^2)^2}\right] = -f(x)$

$\int_{-2}^2 \frac{3x}{(9+x^2)^2} dx = 0$, because $\frac{3x}{(9+x^2)^2}$ is an odd function.

b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (x^2 + \sec^2 x) dx = 2 \int_0^{\frac{\pi}{4}} (x^2 + \sec^2 x) dx,$

because $f(x) = x^2 + \sec^2 x$ is an even function.

$$= 2 \left[\frac{x^3}{3} + \tan x \right]_0^{\frac{\pi}{4}} = 2 \left[\frac{\pi^3}{3(64)} + 1 - 0 - 0 \right] = \frac{\pi^3}{96} + 2$$

Def. (Areas Between Curves)

If f and g are continuous with $f(x) \geq g(x)$ through $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is defined as, (Figure 5.14.a)

$$A = \int_a^b [f(x) - g(x)] dx$$

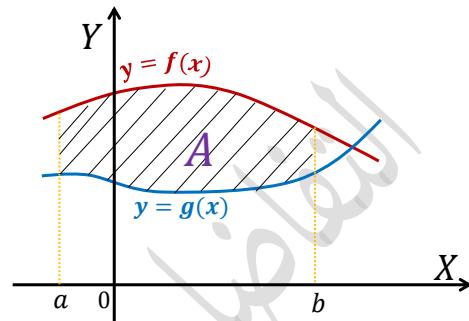


Figure 5.14.a The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$

Similarly, (Figure 5.14.b)

$$A = \int_c^d [f(y) - g(y)] dy$$

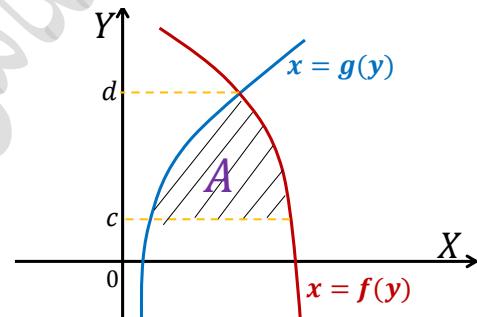


Figure 5.14.b The region between the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$

Ex.(4) Find the areas of the regions enclosed by the following curves and lines.

a) $y = x^2 - 2$ and $y = 2$. Figure 5.15

Sol. $x^2 - 2 = 2 \rightarrow x^2 = 4 \rightarrow x = \pm 2$

$$A = \int_{-2}^2 [2 - (x^2 - 2)] dx$$

$$= \int_{-2}^2 (4 - x^2) dx$$

$$= 2 \int_0^2 (4 - x^2) dx$$

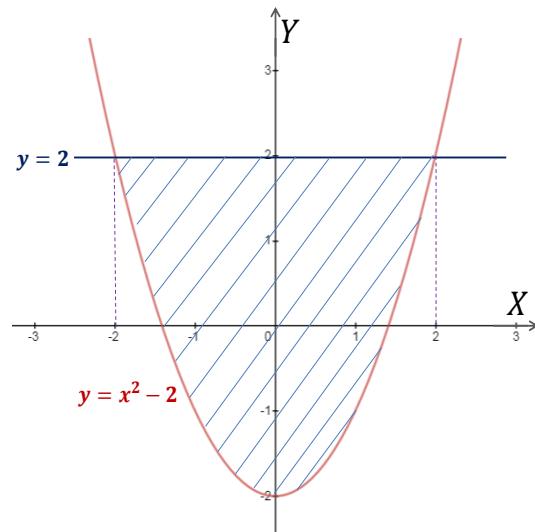


Figure 5.15 The region between $y = x^2 - 2$ and $y = 2$

$$= 2 \left[4x - \frac{x^3}{3} \right]_0^2 = 2 \left[8 - \frac{8}{3} - 0 \right] = 2 \left(\frac{16}{3} \right) = \boxed{\frac{32}{3}}$$

b) $y = 4 - x^2$, $y = 2 - x$, $x = -2$ and $x = 2$

Sol. Figure 5.16

$$4 - x^2 = 2 - x$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2, x = -1$$

$$A_1 = \int_{-2}^{-1} [2 - x - (4 - x^2)] dx$$

$$= \int_{-2}^{-1} (2 - x - 4 + x^2) dx$$

$$= \int_{-2}^{-1} (x^2 - x - 2) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-2}^{-1}$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \left[-\frac{8}{3} - 2 + 4 \right]$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 + \frac{8}{3} + 2 - 4 = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \boxed{\frac{11}{6}}$$

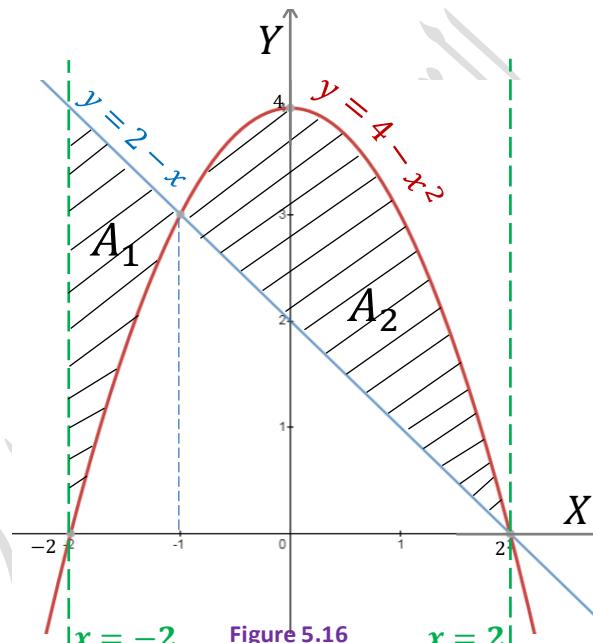
$$A_2 = \int_{-1}^2 [4 - x^2 - (2 - x)] dx = \int_{-1}^2 (4 - x^2 - 2 + x) dx$$

$$= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2$$

$$= 4 + 2 - \frac{8}{3} - \left[-2 + \frac{1}{2} + \frac{1}{3} \right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

$$= 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \boxed{\frac{9}{2}}$$

$$A = A_1 + A_2 = \frac{11}{6} + \frac{9}{2} = \boxed{\frac{19}{3}}$$



Ex.(5) Find the area of region enclosed by the following curves.

a) $x = y^2$ and $x + 2y^2 = 3$.

$$\rightarrow x = -2y^2 + 3$$

Sol. Figure 5.18

$$y^2 = -2y^2 + 3 \rightarrow 3y^2 = 3$$

$$y^2 = 1 \rightarrow y = \pm 1$$

$$A = \int_{-1}^1 (-2y^2 + 3 - y^2) dy$$

$$= \int_{-1}^1 (3 - 3y^2) dy$$

$$= [3y - y^3]_{-1}^1$$

$$= 3 - 1 - (-3 + 1)$$

$$= 3 - 1 + 3 - 1 = \boxed{4}$$

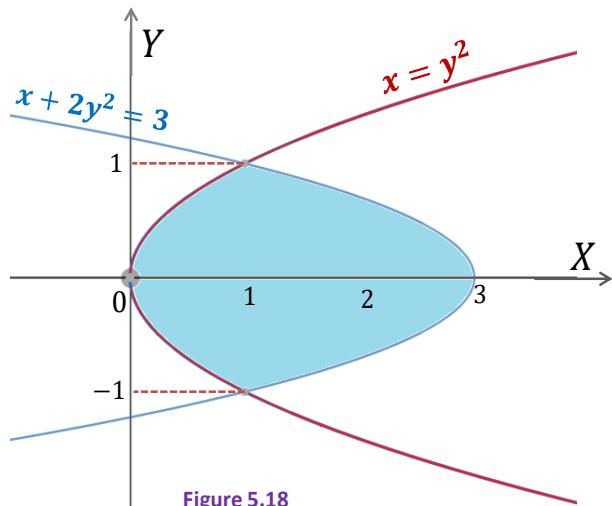


Figure 5.18

Exercises 5.5:

A. Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form.

1) $\int \sin 3x \, dx, u = 3x$

2) $\int \sec 2t \tan 2t \, dt, u = 2t$

Sol(1). Let $u = 3x \rightarrow du = 3dx \rightarrow \frac{1}{3}du = dx$

$$\int \sin 3x \, dx = \int \frac{1}{3} \sin u \, du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos 3x + C$$

3) $\int 28(7x - 2)^{-5} \, dx, u = 7x - 2$ 4) $\int \frac{9r^2 dr}{\sqrt{1-r^3}}, u = 1 - r^3$

5) $\int \csc^2 2\theta \cot 2\theta \, d\theta,$ $\begin{cases} a) \text{Using: } u = \cot 2\theta \\ b) \text{Using: } u = \csc 2\theta \end{cases}$

6) $\int \sqrt{x} \sin^2(x^{\frac{3}{2}} - 1) \, dx, u = x^{\frac{3}{2}} - 1$

B. Evaluate the following integrals.

7) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx$

8) $\int \sqrt{3-2s} \, ds$

9) $\int (s^3 + 2s^2 - 5s + 5)(3s^2 + 4s - 5) \, ds$

10) $\int r^2 \left(\frac{r^3}{18} - 1\right) \, dr$

11) $\int x^3 \sqrt{x^2 + 1} \, dx$

12) $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} \, dt$

C. Use the Substitution to evaluate the integrals in the following :

13) $\int_{-1}^1 r \sqrt{1-r^2} \, dr$

14) $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

15) $\int_{2\pi}^{3\pi} 3 \cos^2 x \sin x \, dx$

16) $\int_0^1 \frac{10 \sqrt{v}}{\left(1+v^{\frac{3}{2}}\right)^2} \, dv$

17) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx$

18) $\int_{-\frac{\pi}{2}}^0 \frac{\sin w}{(3 + 2 \cos w)^2} \, dw$

19) $\int_1^4 \frac{dy}{2 \sqrt{y} (1 + \sqrt{y})^2}$

20) $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-\frac{2}{3}} (12y^2 - 2y + 4) dy$

Answer: 13)0 , 14) $\frac{1}{2}$, 15)2 , 16) $\frac{10}{3}$, 17)0 , 18) $-\frac{1}{15}$, 19) $\frac{1}{6}$, 20)3

التفاضل والتكامل 2 . د. ملاذ رحيم

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