

جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل 2

للطلبة المرحلة الاولى

الفصل السابع: المحاضرة السابعة

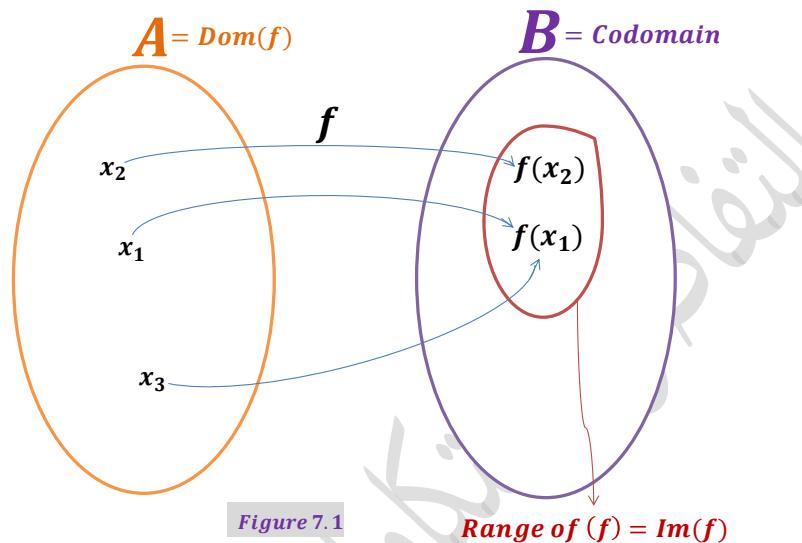
(معكوس الدوال ومشتقاتها)

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CH. 7: Transcendental Functions

7.1 Inverse Function and Their Derivatives

Def. The function f is a rule that assigns to each element x in a set A a unique element $f(x)$ in the set B .



Def. A function $f: D \rightarrow R$ is called one-to-one if not different element of D have same Image in R , that is

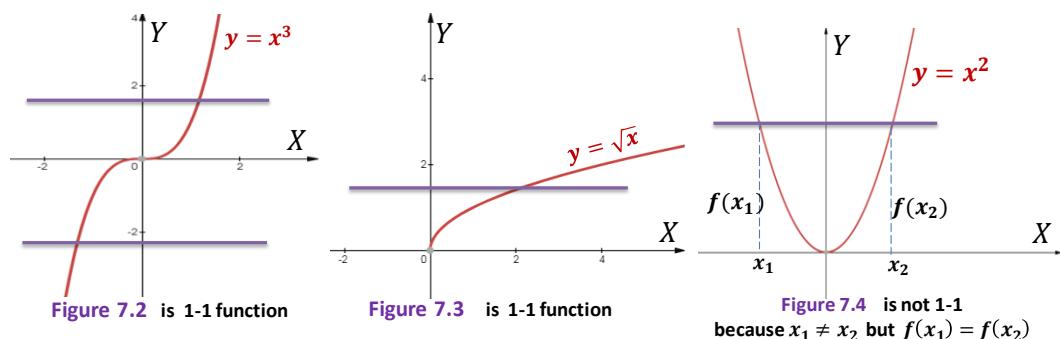
if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, $x_1, x_2 \in D$

Equivalently,

$f: D \rightarrow R$ is 1 – 1 if $f(x_1) = f(x_2)$ then $x_1 = x_2$, $x_1, x_2 \in D$

Def. A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

For example, the functions $y = x^3$ (**Figure 7.2**) and $y = \sqrt{x}$ (**Figure 7.3**) are 1 – 1 but the function $y = x^2$ (**Figure 7.4**) is not 1 – 1 because of the horizontal line test.



Remark. Any increasing (Figure 7.5) or decreasing function (Figure 7.6) is 1 – 1 .

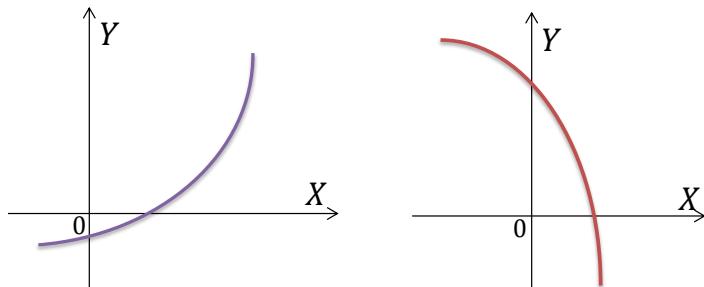


Figure 7.5 Increasing Function

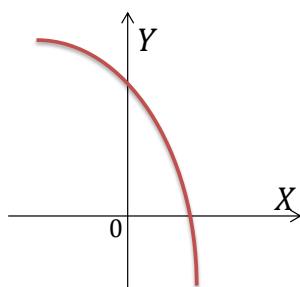


Figure 7.6 Decreasing Function

Ex.(1) Are the function $f(x) = (x + 1)^3 + 2$ and $g(x) = x + \frac{1}{x}$ one-to-one?

Sol. Assume that $f(x_1) = f(x_2) \Rightarrow (x_1 + 1)^3 + 2 = (x_2 + 1)^3 + 2$

$$\xrightarrow{\text{By adding } (-2) \text{ to both sides}} (x_1 + 1)^3 = (x_2 + 1)^3 \xrightarrow{\text{By taking } \sqrt[3]{} } x_1 + 1 = x_2 + 1$$

$$\xrightarrow{\text{By adding } (-1) \text{ to both sides}} x_1 = x_2 \Rightarrow f \text{ is 1-1.}$$

$$\text{And } g(x) = x + \frac{1}{x} \Rightarrow g(2) = 2 + \frac{1}{2} = \frac{5}{2} \text{ and } g\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = \frac{5}{2}$$

So, g is not 1 – 1 because $g(2) = g\left(\frac{1}{2}\right)$ but $2 \neq \frac{1}{2}$.

Def. Suppose that f is a one-to-one function on a domain D with range R . The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b$$

* The domain of f^{-1} is R and the range of f^{-1} is D . (Figure 7.7)

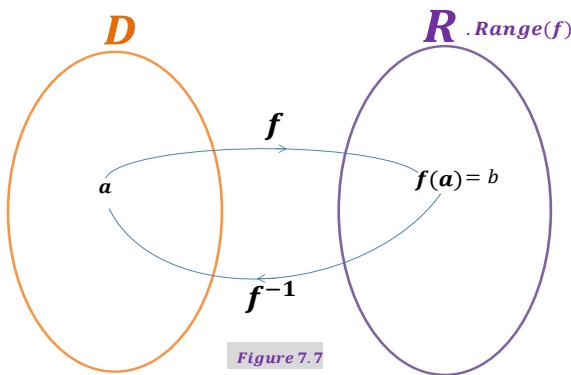


Figure 7.7

Properties of f^{-1}

1) If f^{-1} is the inverse function of f then $f^{-1} \neq \frac{1}{f}$

2) If $f(a) = b \rightarrow f^{-1}(f(a)) = f^{-1}(b) \rightarrow (f^{-1} \circ f)(a) = f^{-1}(b)$

$$\rightarrow a = f^{-1}(b)$$

3) $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x , \quad x \in Dom(f)$

4) $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x , \quad x \in Dom(f^{-1})$

5) f has an inverse f^{-1} if and only if f is 1 – 1

Ex.(2) Let $f(x) = 8x^3 + 3$ show that:

i) f^{-1} exists, ii) show that $f^{-1}(x) = \frac{1}{2} \sqrt[3]{x-3}$

Sol. i) Assume that $f(x_1) = f(x_2)$, then $8x_1^3 + 3 = 8x_2^3 + 3$

$\rightarrow 8x_1^3 = 8x_2^3 \rightarrow x_1^3 = x_2^3 \rightarrow x_1 = x_2$ so f is 1 – 1 $\rightarrow f^{-1}$ exists

Another sol. $f'(x) = 24x^2 \geq 0 \quad \forall x \rightarrow f$ is increasing $\rightarrow f$ is 1 – 1

$$\text{ii)} f^{-1}(f(x)) = f^{-1}(8x^3 + 3) = \frac{1}{2} \sqrt[3]{8x^3 + 3 - 3}$$

$$= \frac{1}{2} \sqrt[3]{8x^3} = \frac{1}{2}(2x) = x$$

$$\text{And } f(f^{-1}(x)) = f\left(\frac{1}{2} \sqrt[3]{x-3}\right) = 8\left(\frac{1}{2} \sqrt[3]{x-3}\right)^3 + 3$$

$$= 8\left(\frac{1}{8}(x-3)\right) + 3 = x - 3 + 3 = x$$

so $\frac{1}{2} \sqrt[3]{x-3}$ is the inverse function of $8x^3 + 3$

Ex.(3) Let $f(x) = x^3 + 2x - 3$, find the value of a if $f^{-1}(a) = 2$.

Sol. $f^{-1}(a) = 2 \rightarrow f(f^{-1}(a)) = f(2) \rightarrow a = f(2)$

$$\rightarrow a = (2)^3 + 2(2) - 3 \rightarrow [a = 9]$$

Ex.(4) Determine whether or not the following functions are 1 – 1. If so, find its inverse.

1) $f(x) = 3x^2 + 1$

Sol. $f(2) = 3(2)^2 + 1 = 13$ and $f(-2) = 3(-2)^2 + 1 = 13$

So, f is not 1 – 1 because $f(2) = f(-2)$ but $2 \neq -2$

$\rightarrow f^{-1}$ does not exists.

Theorem 7.1- The Derivative Rule for Inverses If $Dom(f)$ is the interval I, $f'(x)$ exists and $f'(x) \neq 0$ on I then f' is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point $b = f(a)$ in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$. That is,

$$\left(\frac{df^{-1}}{dx}\right)_{x=b=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=a=f^{-1}(b)}}$$

Ex.(5) If $f(x) = x^2 - 4x - 5$, $x > 2$, find the value of $\frac{df^{-1}}{dx}$

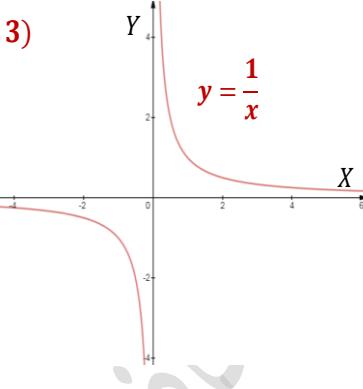
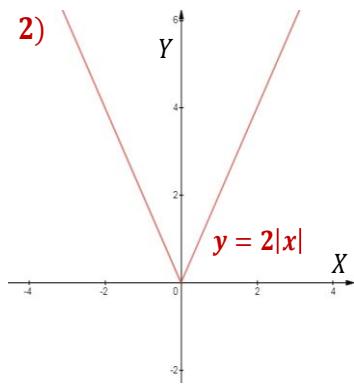
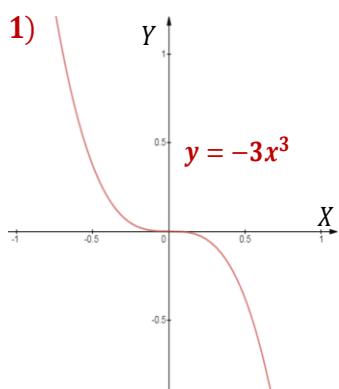
at $x = 0 = f(5)$. 0 in $Dom(f^{-1})$, 5 in $Dom(f)$

Sol.

$$\left(\frac{df^{-1}}{dx}\right)_{x=0=f(5)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=5}} = \frac{1}{(2x-4)_{x=5}} = \frac{1}{(2(5)-4)} = \frac{1}{6}$$

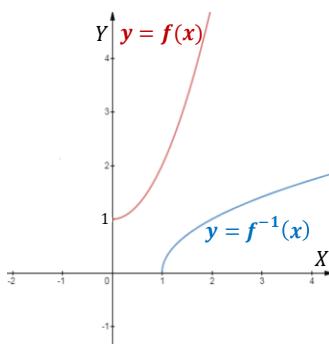
Exercises 7.1:

A. Which of the functions graphed in Exercises 1-3 are one-to-one, and which are not?

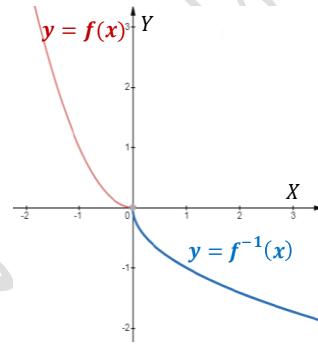


B. Exercises 4-6 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1} . Find a formula for f^{-1} in each case.

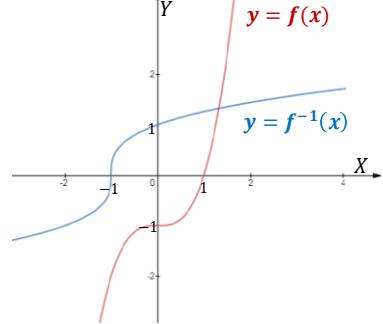
4) $f(x) = x^2 + 1, x \geq 0$



5) $f(x) = x^2, x \leq 0$



6) $f(x) = x^3 - 1$



C. Each of Exercises 7-9 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1} . As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

7) $f(x) = x^5$

Sol. i) $y = x^5 \rightarrow x = y^{\frac{1}{5}}$

ii) $y = \sqrt[5]{x} = f^{-1}(x)$

$\text{Dom}(f^{-1}) = R, \text{Rng}(f^{-1}) = R$

$f(f^{-1}(x)) = \left(x^{\frac{1}{5}}\right)^5 = x \text{ and } f^{-1}(f(x)) = (x^5)^{\frac{1}{5}} = x$

8) $f(x) = x^4, x \geq 0$

9) $f(x) = x^3 + 1$

D. a) Find $f^{-1}(x)$. b) Evaluate $\frac{df}{dx}$ at $x = a$ and $\frac{df^{-1}}{dx}$ at $x = f(a)$ to show that at these points $\frac{df^{-1}}{dx} = \frac{1}{\frac{df}{dx}}$.

10) $f(x) = \frac{1}{5}x + 7, a = -1$

المصادر:

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