جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

المادة: التفاضل والتكامل2

للطلبة المرحلة الاولى

الفصل السابع: المحاضرة التاسعة

(الدوال الاسية)

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CH. 7: Transcendental Functions

7.3 The Exponential Function

* Since $\ln x$ is an increasing function $\rightarrow \ln x \ is\ 1-1 \rightarrow \ln^{-1} x$ exist.

* $\ln e = 1 \rightarrow \ln^{-1}(\ln e) = \ln^{-1}(1) \rightarrow e = \ln^{-1} 1 \approx 2.7$

$\boxed{e = 2.71828182846}$

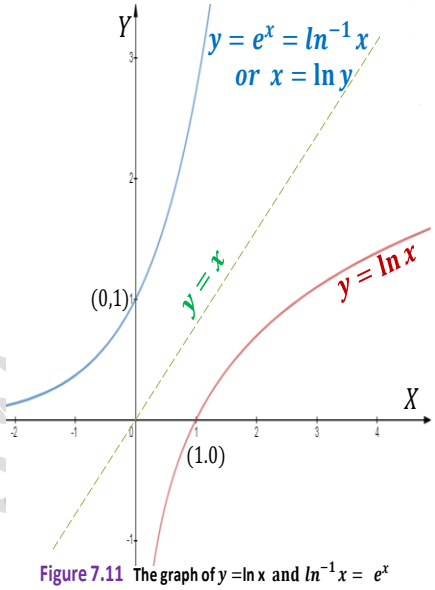

Figure 7.11 The graph of $y = \ln x$ and $ln^{-1} x = e^x$

Def. For every real number x, we define the natural exponential function to be the inverse of $\ln x$. That is $exp(x) = e^x = \ln^{-1} x$.

Some Properties of $y = e^x$

1) $Dom(e^x) = Rng(\ln x) = (-\infty, \infty)$ and $Rng(e^x) = Dom(\ln x) = (0, \infty)$.

2) $e^x > 0, \forall x$.

3) $e^{\ln x} = \ln^{-1}(\ln x) \rightarrow e^{\ln x} = x, x > 0$

and $\ln e^x = x \ln e \rightarrow \ln e^x = x, \forall x$

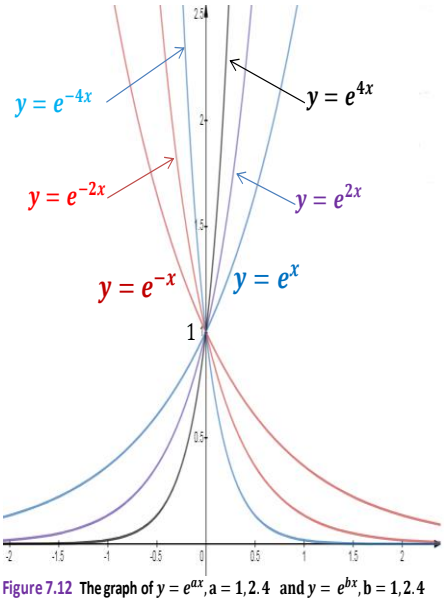

Figure 7.12 The graph of $y = e^{ax}$, a = 1, 2. 4 and $y = e^{bx}$, b = 1, 2. 4

4) i. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$, ii. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$, iii. $e^{-x} = \frac{1}{e^x}$, iv. $(e^{x_1})^{x_2} = e^{x_1 \cdot x_2}$.

5) $\lim_{x\to\infty} e^x = \infty, \lim_{x\to-\infty} e^x = 0$

$\boxed{\lim_{x\to 0} e^x = 1}$

6) $\frac{d}{dx}(e^x) = e^x$

Proof. Let $y = e^x \rightarrow \ln y = \ln e^x \rightarrow \ln y = x \rightarrow \frac{1}{y} y' = 1 \rightarrow y' = y \rightarrow y' = e^x$.

If u is a differentiable function of x then $\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$

For example $\frac{d}{dx}\left(e^{3x^2}\right) = e^{3x^2} \cdot (6x)$.

7) $\int e^x\,dx = e^x + C, \quad \int e^{ax}\,dx = \frac{e^{ax}}{a} + C.$

Theorem 7.2

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y$$

Ex.(1) **Solve the equation $e^{2x-6} = 4$ for x.**

Sol. $\ln e^{2x-6} = \ln 4 \to 2x - 6 = \ln 4 \to 2x = 6 + \ln 4 \to x = 3 + \frac{1}{2}\ln 4$

$$x = 3 + \ln(2^2)^{\frac{1}{2}} \to x = 3 + \ln 2$$

Ex.(2) $\int_0^{\ln 2} e^{3x}\,dx \overset{Sol.}{\Longrightarrow} = \frac{1}{3}\int_0^{\ln 2} e^{3x}\,3dx = \frac{1}{3}[e^{3x}]_0^{\ln 2} = \frac{1}{3}\left[e^{3\ln 2} - e^{3(0)}\right]$

$$= \frac{1}{3}\left[e^{\ln 2^3} - e^0\right] = \frac{1}{3}(8-1) = \frac{7}{3}$$

The General Exponential Function a^x

Def. For any real numbers $a > 0$ and x, the exponential function with base a is defined as $a^x = e^{x\ln a}$.

Def. For any $x > 0$ and for any real number $n, x^n = e^{n\ln x}$. $\boxed{x^x = e^{x\ln x}}$

Properties of the function $y = a^x$

1) $Dom(a^x) = (-\infty, \infty)$ and

$Rng(a^x) = (0, \infty)$

2) $a^x > 0\,,\ \forall x$

3) i. $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$, ii. $\frac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$,

iii. $a^{-x} = \frac{1}{a^x}$, iv. $(a^{x_1})^{x_2} = a^{x_1\cdot x_2}$.

4) $y = a^x$, then $y' = a^x \ln a$

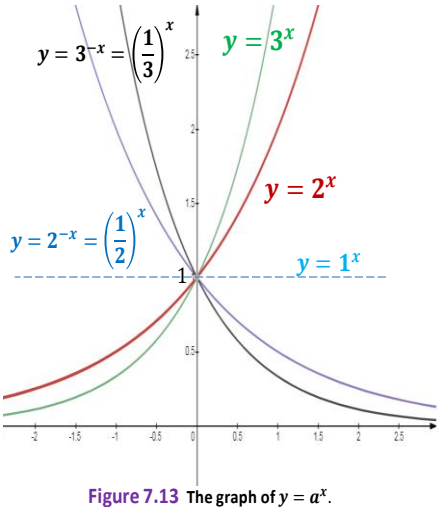

Figure 7.13 The graph of $y = a^x$.

$$\boxed{\begin{array}{c} Show\ why\ y = e^x \to y' = e^x? \\ when\ a = e, y = e^x \to y' = e^x \ln e = e^x \end{array}}$$

$$\boxed{\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\frac{d}{dx}(x\ln a) = e^{\ln a^x}(\ln a) = a^x \ln a}$$

If u is a differentiable function of x, then $y = a^u \to y' = a^u \ln a \frac{du}{dx}$.

Def. For any positive real number $a \neq 1$, then function $\log_a x$ is the inverse function of a^x. That is, if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.

Properties of $y = \log_a x$

1)$(f^{-1}{}_o f)(x) = x \to f^{-1}\big(f(x)\big) = x \to f^{-1}(a^x) = x,$

$\to \log_a(a^x) = x, \forall x \to x\log_a a = x\} \div x \to \boxed{\log_a a = 1}$.

2)$(f_o f^{-1})(x) = x \to f\big(f^{-1}(x)\big) = f(\log_a x) = \boxed{a^{\log_a x} = x,\ \ x > 0}$.

3)$\log_a x = \frac{\ln x}{\ln a}$, $\boxed{a \neq 1\ since\ \log_1 x = \frac{\ln x}{\ln 1} = \frac{\ln x}{0}?}$

Proof. $a^{\log_a x} = x \to \ln\left(a^{\log_a x}\right) = \ln x \to \log_a x \ln a = \ln x$

$\to \log_a x = \frac{\ln x}{\ln a}$.

4)$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{x\ln a}$

If u is a differentiable function of x, then $\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\cdot\frac{du}{dx}$.

5)i.$\log_a(xy) = \log_a x + \log_a y$, ii. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$,

iii.$\log_a\left(\frac{1}{x}\right) = -\log_a x$, iv.$\log_a x^n = n\log_a x$.

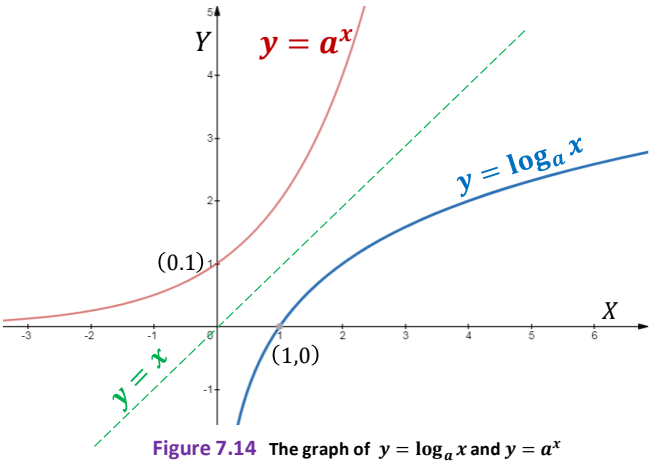

Figure 7.14 The graph of $y = \log_a x$ and $y = a^x$

Ex.(3) **Simplify the following expression**

a) $e^{3\ln 2} + \ln\sqrt[3]{e^9} = e^{\ln 2^3} + \ln(e^9)^{\frac{1}{3}} = 2^3 + \ln e^3 = 8 + 3\ln e = 11$

b) $\ln e^2 + \log_3 9 - \log_2\sqrt{8} = 2\ln e + \log_3 3^2 - \log_2(2^3)^{\frac{1}{2}}$

$= 2 + 2\log_3 3 - \frac{3}{2}\log_2 2 = 2 + 2 - \frac{3}{2} = 4 - \frac{3}{2} = \frac{5}{2}$

Ex.(4) **Solve the equation for x:** $3^{\log_3 x^2} = 5e^{\ln x} - 3\cdot 10^{\log_{10} 2}$

Sol. $\rightarrow x^2 = 5(x) - 3(2) \rightarrow x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0 \rightarrow x = 3 \; or \; x = 2$

Ex.(5) **Find y' for each the following**

1) $y = e^{5-7x^2} \overset{Sol.}{\Longrightarrow} y' = e^{5-7x^2}\cdot(-14x) = -14xe^{5-7x^2}$

2) $y = e^{\sin x}(\ln(x^2+1))$

Sol. $y' = e^{\sin x}\cdot\frac{1}{x^2+1}(2x) + \ln(x^2+1)\cdot e^{\sin x}\cdot\cos x$

Ex.(6) Evaluate the following integrals.

1) $\int \frac{e^{-\frac{1}{x^2}}}{x^3}dx$, $t = -\frac{1}{x^2} \rightarrow dt = \frac{2}{x^3} \rightarrow \frac{1}{2}dt = \frac{1}{x^3}dx$

Sol. $= \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + C = \frac{1}{2}e^{-\frac{1}{x^2}} + C$

2) $\int_0^{\frac{\pi}{4}} \left(1+e^{\tan\theta}\right)\sec^2\theta\, d\theta$, $w=\tan\theta \to dw=\sec^2\theta\, d\theta$

$$=\int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta+\int_0^{\frac{\pi}{4}} e^{\tan\theta}\sec^2\theta\, d\theta=\int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta+\int_0^{1} e^{w}\, dw$$

$$=[\tan\theta]_0^{\frac{\pi}{4}}+[e^w]_0^1=\left[\tan\frac{\pi}{4}-\tan 0\right]+[e^1-e^0]$$

$$=(1-0)+(e-1)=1+e-1=e$$

3) $\int 7^x\, dx=\frac{7^x}{\ln 7}+c.$ **4)** $\int (1.4)^x\, dx=\frac{(1.4)^x}{\ln(1.4)}+c$

5) $\int_1^2 \frac{2^{\ln x}}{x}\, dx=\int_1^2 2^{\ln x}\frac{1}{x}dx,$ $z=\ln x \to dz=\frac{1}{x}dx$

$$=\int_0^{\ln 2} 2^z\, dz=\left[\frac{2^z}{\ln 2}\right]_0^{\ln 2}=\frac{1}{\ln 2}\left(2^{\ln 2}-2^0\right)=\frac{2^{\ln 2}-1}{\ln 2}$$

المصادر:

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