جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

Planar Graph / Part 1

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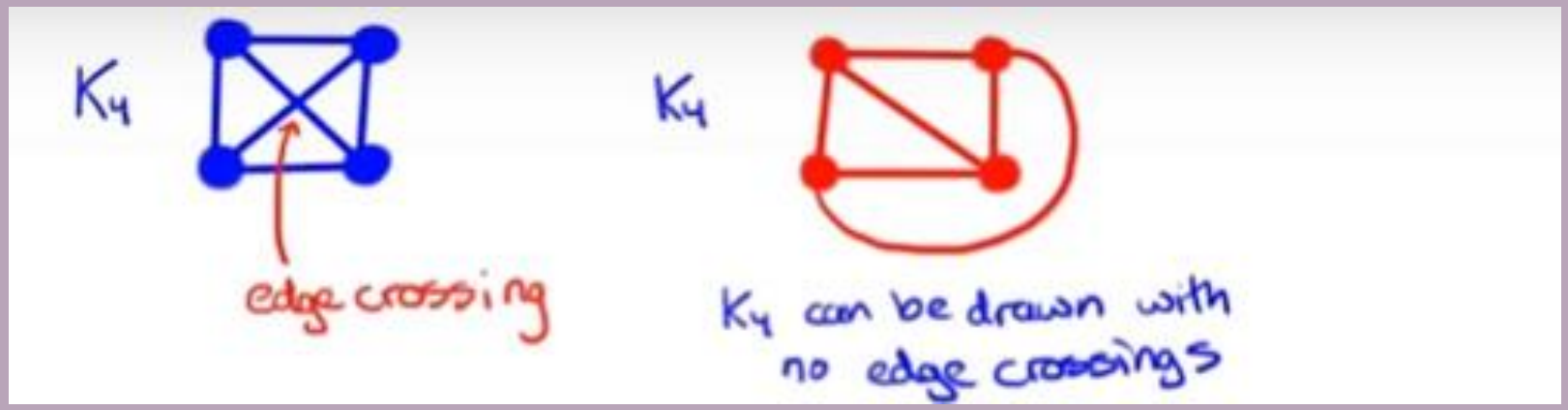
Planar Graphs

Lecture #1

Part #1

A graph is called ***planar*** if it can be drawn on a plane without edges crossing.

A ***plane*** graph is a planar graph has been drawn in the plane without edges crossing.



The graph in blue color is ***planar graph*** , but the graph in read color is ***plane graph***.

Here are two ways to determine if a graph is planar:

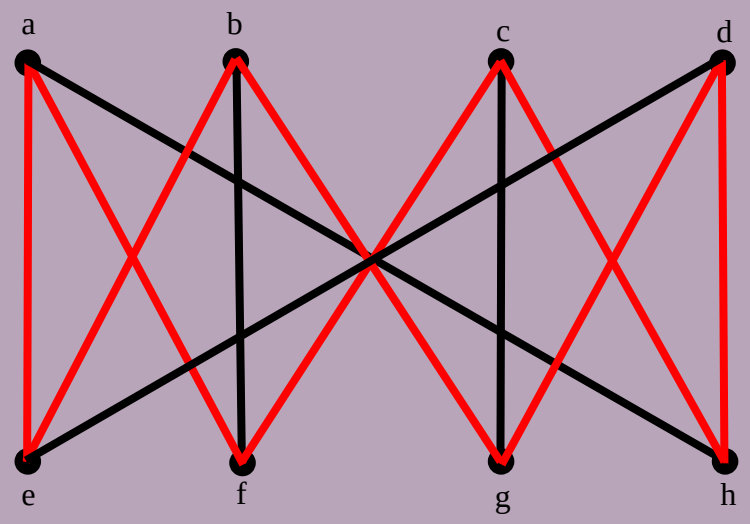
1) ***Circle-Chord Method:*** It consists of a step-by-step method of drawing the graph, edge-by-edge without crossing any edges.

2) Theoretical results, such as ***Euler's Theorem (Formula)*** and its consequences or ***Kuratowski's Theorem***.

Circle-Chord Method (Planar)

Step One: Find a **circuit** that contains all the vertices of the graph. (a circuit is a path that ends where it starts)

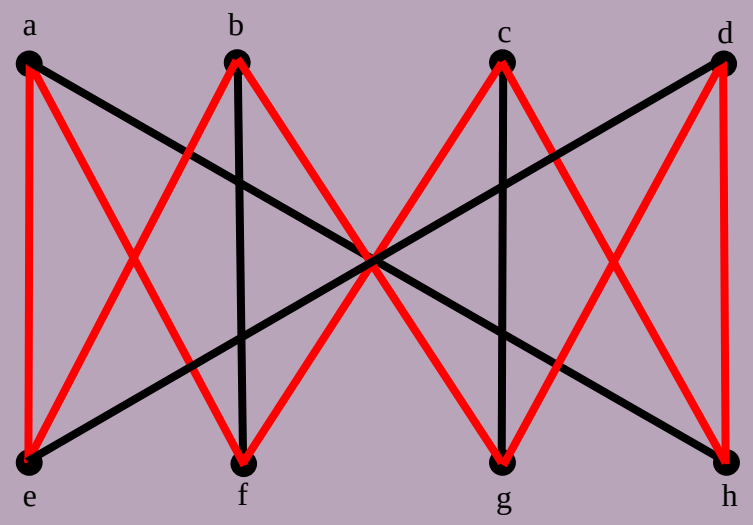
N**ote:** Step one is not always possible, and is often difficult.

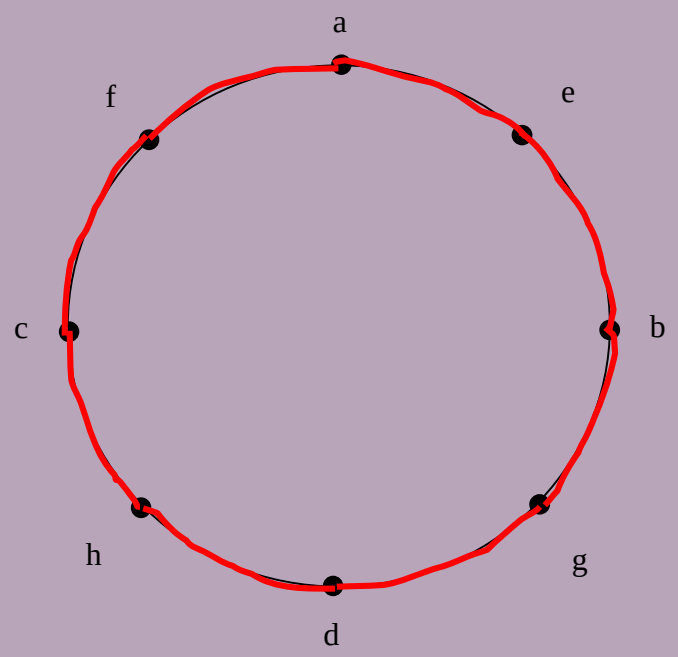


The **circuit** for this graph is highlighted in **red** **.**

Circle-Chord Method (Planar)

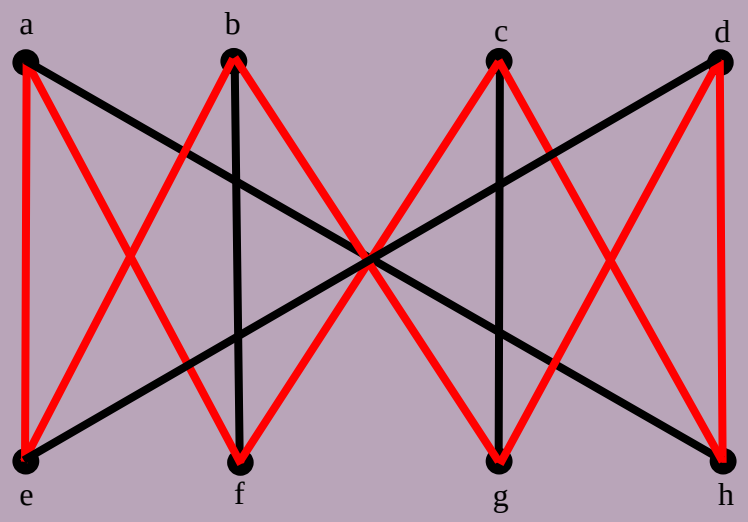
Step Two: Draw this circuit as a large circle.

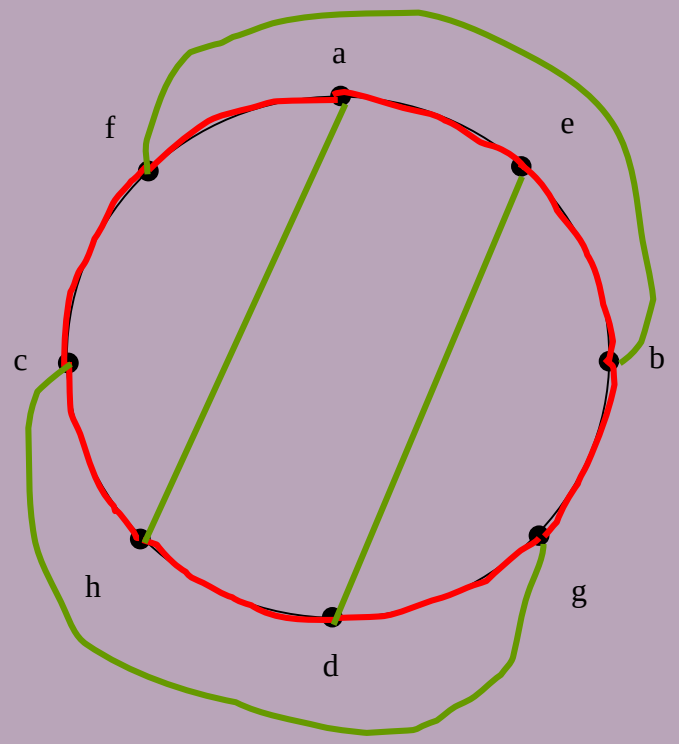




Circle-Chord Method (Planar)

Step Three: Choose one ***chord*** **(**remaining edges in black) , and decide to draw it either inside or outside the circle. If chosen correctly, it will force certain other chords to be drawn opposite to the circle. (Inside if the first chord was drawn outside, and vice versa.)



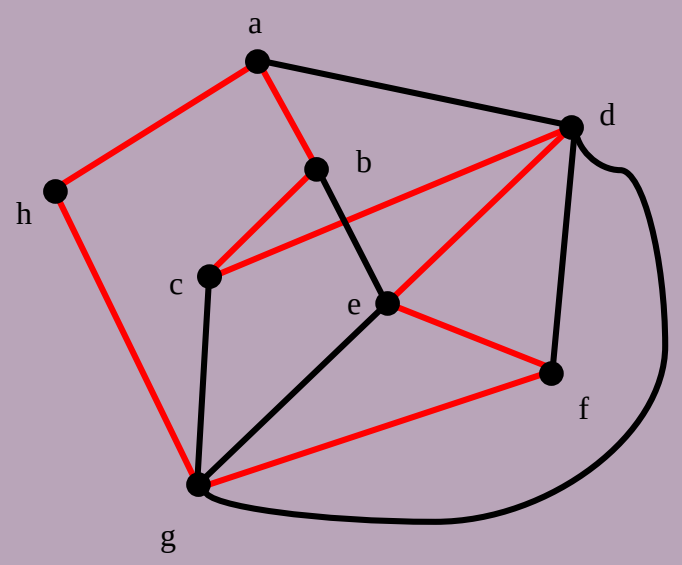


Since the chords could be drawn without crossing, this graph is planar.

Circle-Chord Method (Nonplanar)

Step One: Find a **circuit** that contains all the vertices of the graph. (a circuit is a path that ends where it starts)

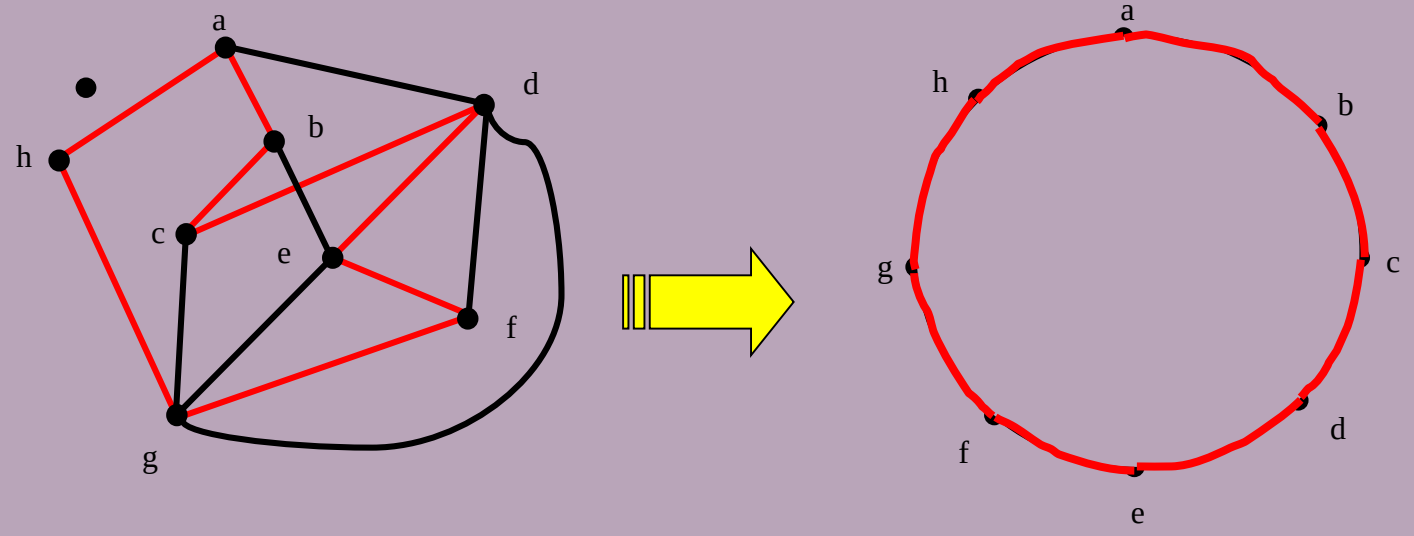
Note: Step one is not always possible, and is often difficult.



The **circuit** for this graph is highlighted in **red.**

Circle-Chord Method (Nonplanar)

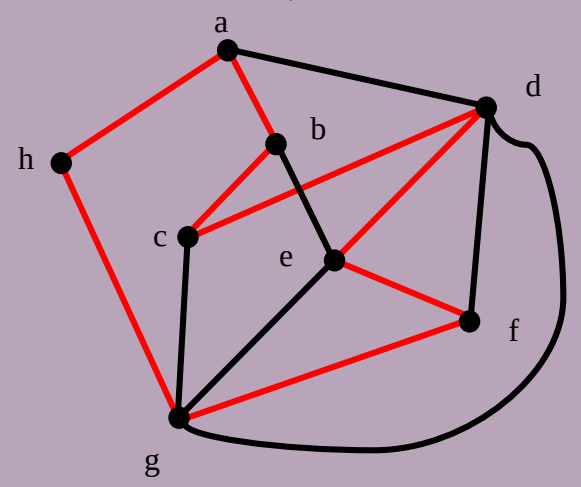
Step Two: Draw this circuit as a large circle.

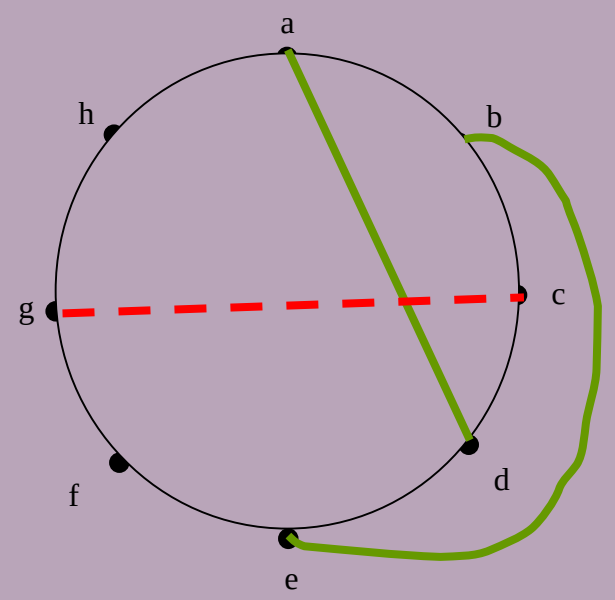


The remaining edges (in black) must be drawn either inside or outside the circle.

Circle-Chord Method--Nonplanar

Step Three: Choose one chord, and decide to draw it either inside or outside the circle. If chosen correctly, it will force certain other chords to be drawn opposite to the circle. (Inside if the first chord was drawn outside, and vice versa.)





Because these lines cross, this is not the plane graph. However, this does not mean that the graph is not planar. Often, it is very difficult to find the correct planar graph.

Kuratowski Graphs and Their Non-planarity:

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called ***Kuratowski's*** graphs and are non-planar.

Notes:

1) K_1, K_2, K_3, K_4 are planar graphs.
2) K_5, K_6, … are non- planar graphs.
3) Bipartite graph $K_{1,n}$, $K_{2,n}$, $K_{n,m}$ $(n < 3\ , m < 3)$ are planar graphs.
4) Bipartite graph $K_{n,m}$ both $n, m \geq 3$ non- planar graphs.

Definition: A face of a graph G is the region formed by a cycle or parallel edges or loops in G.

Theorem (Euler Theorem on Plane Graphs): If G is a connected plane graph, then $|V(G)| + |F(G)| - |E(G)| = 2$, where V, E, F are respectively the vertex set, edge set and set of faces of G.

Theorem 1: If G is a planar graph without parallel edges with n vertices and e edges , where $e \geq 1$, then $e \leq 3n - 6$. Moreover, if G is bipartite graph, then $e \leq 3n - 4$.

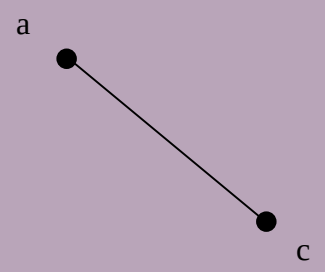
By using Theorem 1, prove K_5 is non-planar graph.

Proof: If possible, let K_5 is planar graph. Then by above theorem $e \leq 3n - 6$. In K_5 , we have $n = 5$, $e = 10$. Hence, $3n - 6 = 9 < e = 10$. Which contradicts the Theorem 1. Hence, K_5 is non-planar graph.

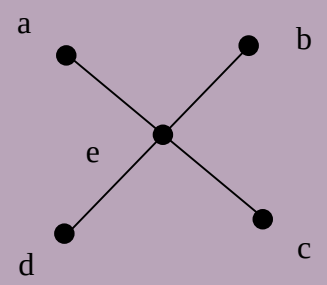
By the same way we can prove $K_{3,3}$ is non-planar graph.

Euler's Theorem (Formula) / Examples

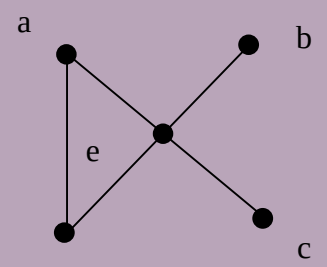
$$|V(G)| + |F(G)| - |E(G)| = 2$$



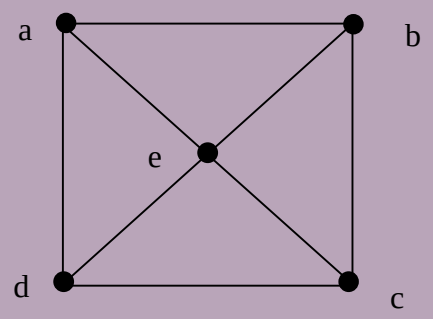
2+1-1=2



5+1-4=2



5+2-5=2



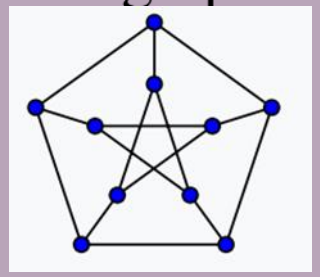
5+5-8=2

مثال يوضح انه اذا تحقق الاتجاه الثاني من المبرهنة الأولى ليس بالضرورة ان يتحقق الاتجاه الأولى وهذا يدل على ان الشرط المذكور في المبرهنة مهم جدا.

For example, $K_{3,3}$ has 6 vertices and 9 edges. So when you substitute into $9 \leq 3(6) - 4$, the equation which holds . However, $K_{3,3}$ in non-planar graph.

Definition: A girth of a graph G is the length of its smallest cycle.
The following theorem is a generalization of Theorem 1.
Theorem 2: Let G be a plane graph with n vertices and let g be the girth of a graph G. Then, $e \leq \frac{g(n-2)}{g-2}$

Example: Using Theorem 2, verify whether the Petersen's graph is planar.



Solution: The smallest cycle is C_5. Therefore, $g = 5$.
If Petersen's graph is planar graph, then by Theorem 2
$e = 15 \leq \frac{g(n-2)}{g-2} = \frac{40}{3}$. Hence, Petersen's graph is non-planar.

Let f be a region of a planar graph G. We define the degree of f, denoted by $d(f)$, as the number of edges on the boundary of f.

Theorem 3: Let G be a plane graph. Then, $\sum_{f \in F(G)} d(f) = 2|E(G)|$.

Example: How many regions would there be in a plane graph with 10 vertices each of degree 3?

Solution: By the first theorem on graph theory (the hand shaking lemma) the sum of the degrees, $10 \times 3 = 30$, equals $2e$, and so $e = 15$. By Euler's formula, the number of regions $|F(G)|$ is:

$$|F(G)| = |E(G)| - |V(G)| + 2 = 15 - 10 + 2 = 7$$

Note: If G contains a non-planar subgraph, then G is non-planar.

Fact: Any subdivision H of a graph G is planar iff G is planar.

Fact: If a graph G is subdivision of K_5 or $K_{3,3}$, then G is non-planar.

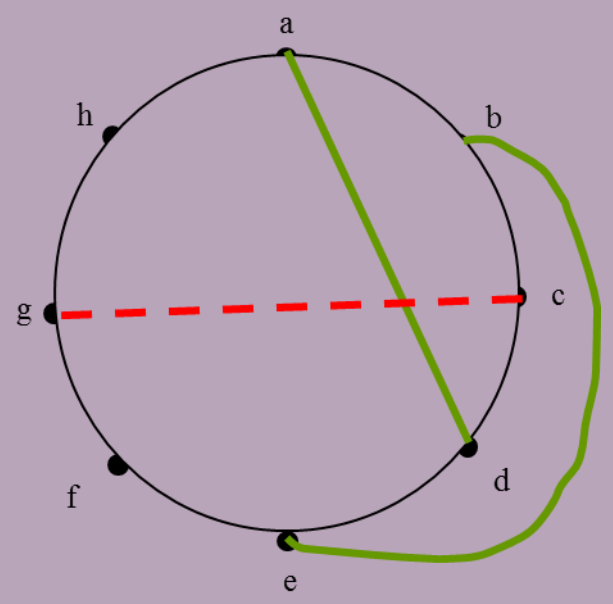
Fact: If a graph G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is non-planar.

Kuratowski's Theorem: A graph G is planar iff has no subdivisions of K_5 and $K_{3,3}$. (In other words, a graph G is planar iff no component of G is homeomorphic to K_5 and $K_{3,3}$.

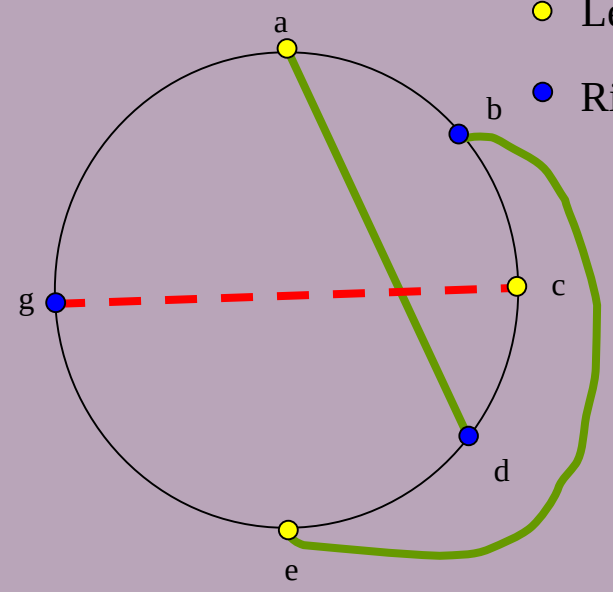
Kuratowski's Theorem (Example)

The $K_{3,3}$ subdivision is more common in non-planar graphs than K_5.

If the graph is non-planar than it contains one of the subdivision with added vertices.



h and f were removed to show the $K_{3,3}$ configuration



THANK YOU

References:

1. S. Arumugam and S. Ramachandran, (2015), Invitation to graph theory, Scitech Publ., Kolkata, India.
2. G. S. Singh, (2013). Graph theory, Prentice Hall of India, New Delhi.
3. R. Balakrishnan and K. Ranganathan, (2012). A textbook of graph theory, Springer, New York.
4. J.A. Bondy and U.S.R Murty, (2008). Graph theory, Springer.
5. G. Agnarsson and R. Greenlaw, (2007). Graph theory: Modeling, applications & algorithms, Pearson Education, New Delhi.
6. G. Chartrand and P. Zhang, (2005). Introduction to graph theory, McGraw-Hill Inc.
7. G. Sethuraman, R. Balakrishnan, and R.J. Wilson, (2004). Graph theory and its applications, Narosa Pub. House, New Delhi.
8. D.B. West, (2001). Introduction to graph theory, Pearson Education Inc., Delhi.
9. V.K. Balakrishnan, (1997). Graph theory, McGrawhill, New York.
10. G. Chartrand and L. Lesniak, (1996). Graphs and digraphs, CRC Press.
11. J.A. Bondy and U.S.R Murty, (1976). Graph theory with applications, North-Holland, New York.