جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

Planar Graph / Part 2

م. د. امين شامان امين

University of Anbar / College of Science / Dept. of Applied Mathematics

Planar Graphs

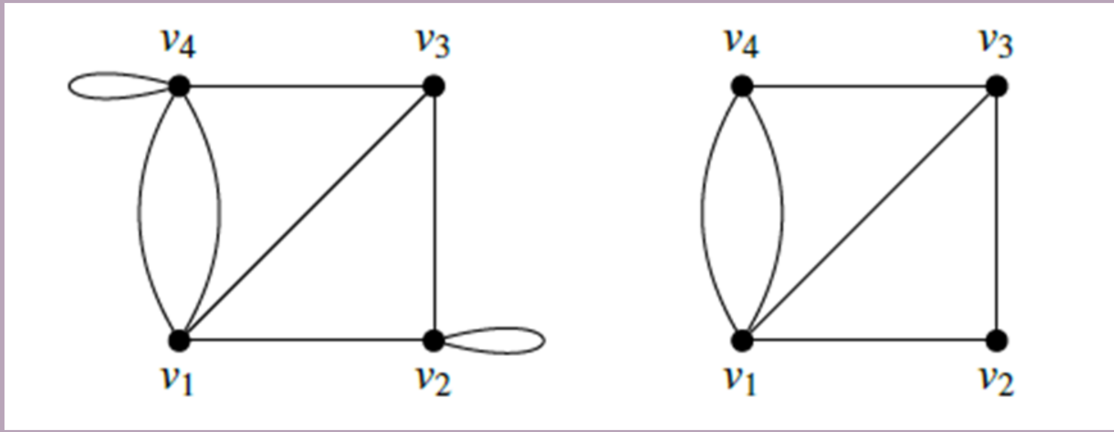
Lecture #1

Part #2

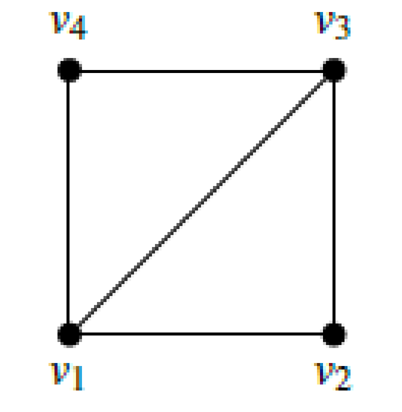
Elementary Reduction in a graph:

Determining whether a given graph is planar by drawing its plane graph may not be a feasible method in all cases. So, a new procedure called elementary topological reduction or simply, an elementary reduction on a given graph to determine whether it is planar. This process has the following steps:

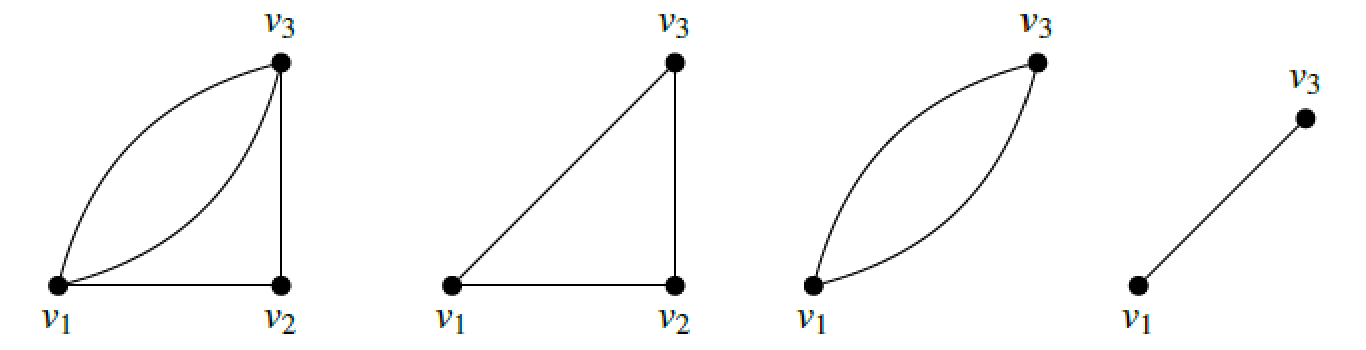
S1: Note that addition and/or removal of self-loops do not affect planarity. So, if G has self-loops, remove all of them.



S2: Similarly, parallel edges do not affect planarity. So, if G has parallel edges, remove all of them, keeping one edge between every pair of vertices.



S3: We observe that removal of vertices having degree 2 by merging the two edges incident on it, perform this action as far as possible.



Repeated performance of these steps will reduce the order and size of the graph without affecting its planarity.

After repeated application of elementary reduction, the given graph will be reduced to any one of the following cases,

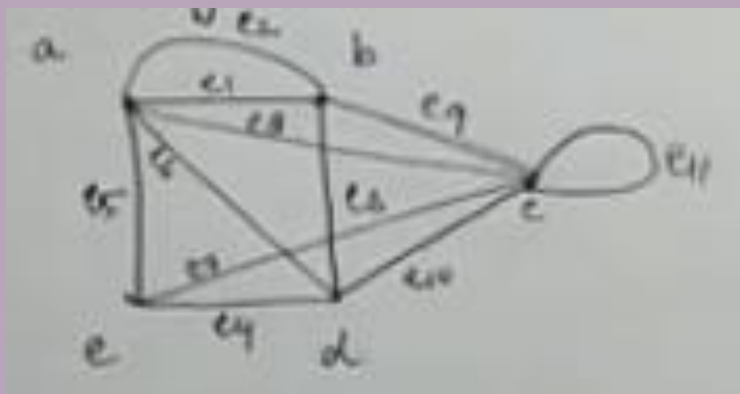
(i) A single edge K_2; or

(ii) A complete graph K_4; or

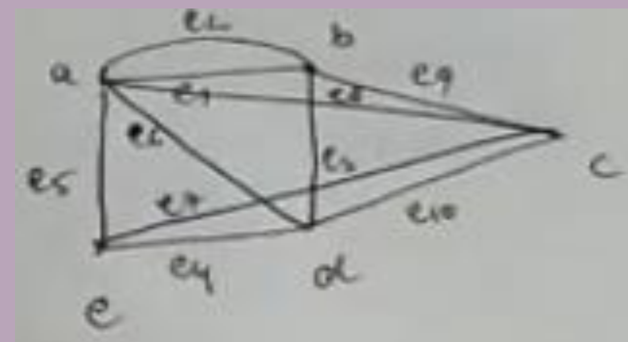
(iii) A non-separable simple graph with $n \geq 5, \varepsilon \geq 7$.

If H is a graph obtained from a graph G by a series of elementary reductions, then G and H are said to be *homeomorphic graphs*. In this case, H is also called a *topological minor* of G.

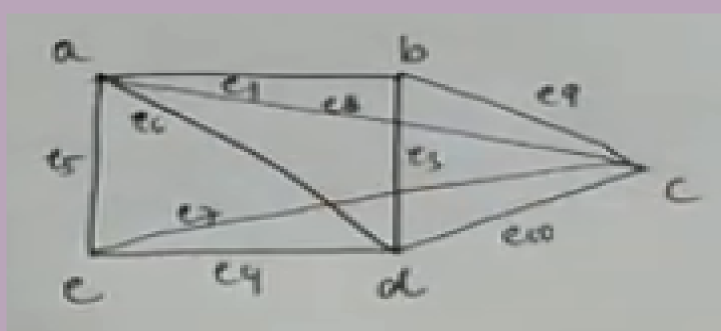
Example: Check whether the following graph planar or not by using Elementary Reduction in a graph.



Step 1: Remove the self –loop.



Step 2: Remove parallel edges. (edge e_2)



Step 3: can not applied because there is no edge of degree 2 .

$$\boldsymbol{n = 5\,, e = 9}$$
$$\boldsymbol{e \le 3n - 6}$$
$$\boldsymbol{9 \le 3(5) - 6 = 9}$$

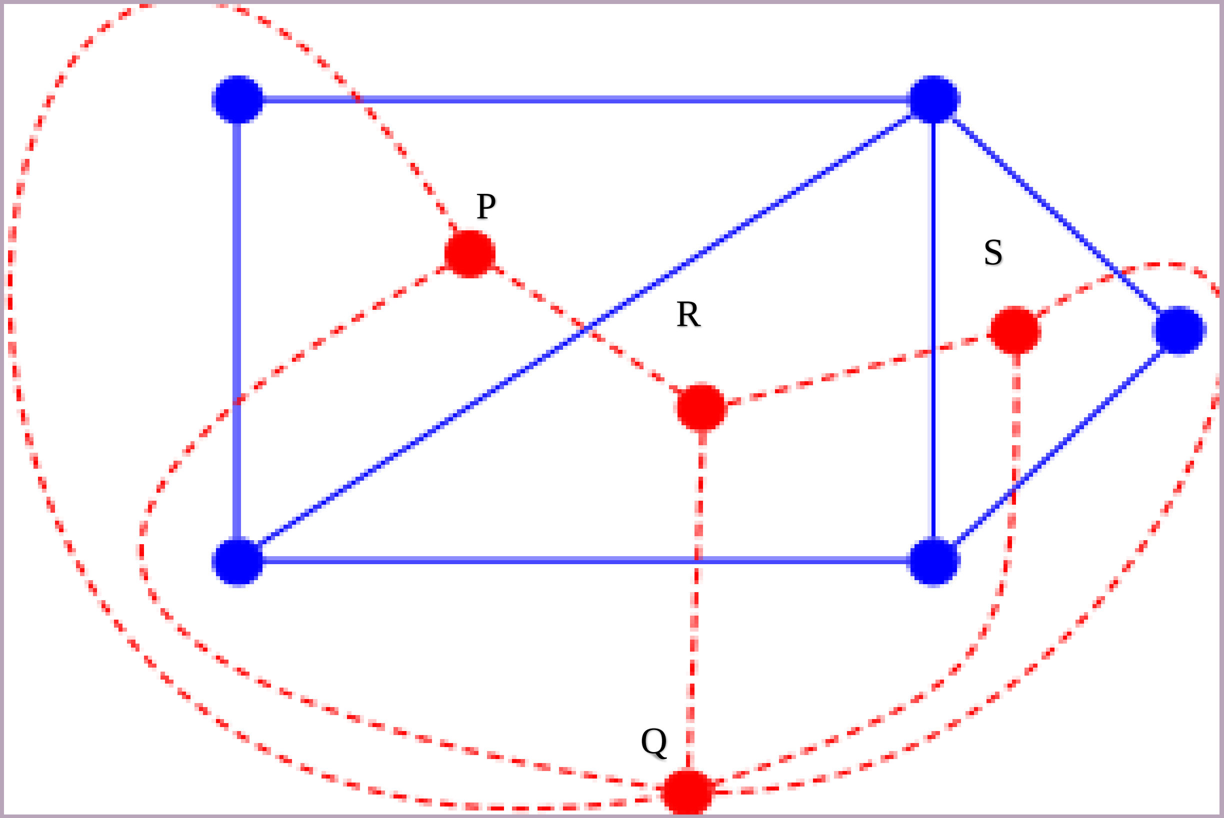
Thus, the graph is planar graph.

Geometric Dual of a Graph

Definition: Given a plane graph G, the dual graph (usually called the geometric dual) of G, denoted by G^*, is the plane graph whose vertices are the faces of G such that two vertices v_i^* and v_j^* in G^* are adjacent in G^* if and only if the corresponding faces f_i and f_j are adjacent in G.

In other words, the correspondence between edges of G and those of the dual G^* is as follows:

If $e \in E(G)$ lies on the boundaries of two faces f_i and f_j in G, then the endpoints of the corresponding dual edge $e^* \in E(G^*)$ are the vertices v_i^* and v_j^* that represent faces f_i and f_j of G.



The red graph is the dual graph of the blue graph.

Note: There will be a one-to-one correspondence between the edges of a graph G and its dual G^* ,one edge of G intersecting one edge of G^*. We note that

$$|V(G^*)| = |F(G)|$$
$$|E(G^*)| = |E(G)|$$
$$|F(G^*)| = |V(G)|$$

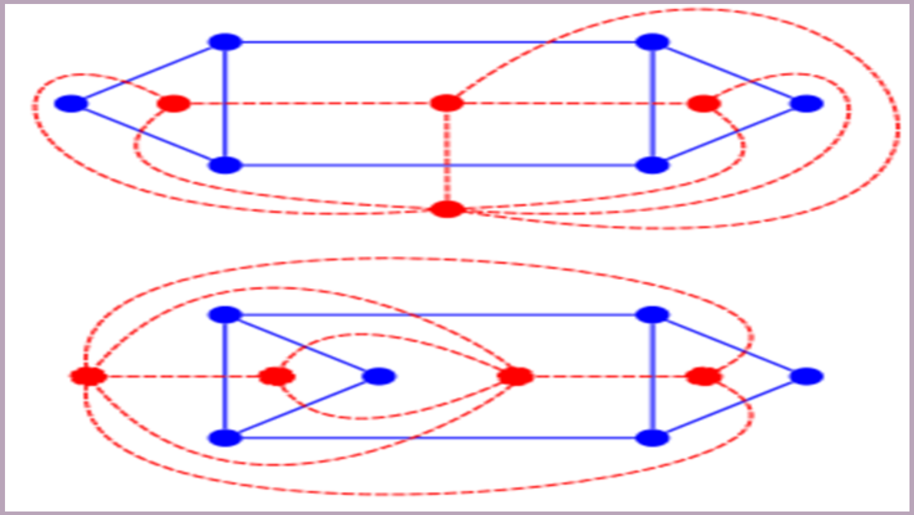
Also, if r and μ respectively denote the rank and nullity of G and r^* and μ^* denote the rank and nullity of G^*, then we observe that $r = \mu^*$ and $\mu = r^*$.

Note: The following observations can be made on the relationships between a planar graph G and its dual G^* :

1) A self-loop in G corresponds to a pendant edge in G^*.
2) A pendant edge in G corresponds to a self-loop in G^*.
3) Edges that are in series in G produce parallel edges in G^*.
4) Parallel edges in G produce edges in series in G^*.
5) The number of edges on the boundary of a face f (the degree of f) in G is equal to the degree of the corresponding vertex v^* in G^*.
6) Both G and G^* are planar.
7) $G^{**} = G$. That is, G is the dual of G^*.

Fact: The two different geometric dual graphs of the same graph (isomorphic) need not be isomorphic.

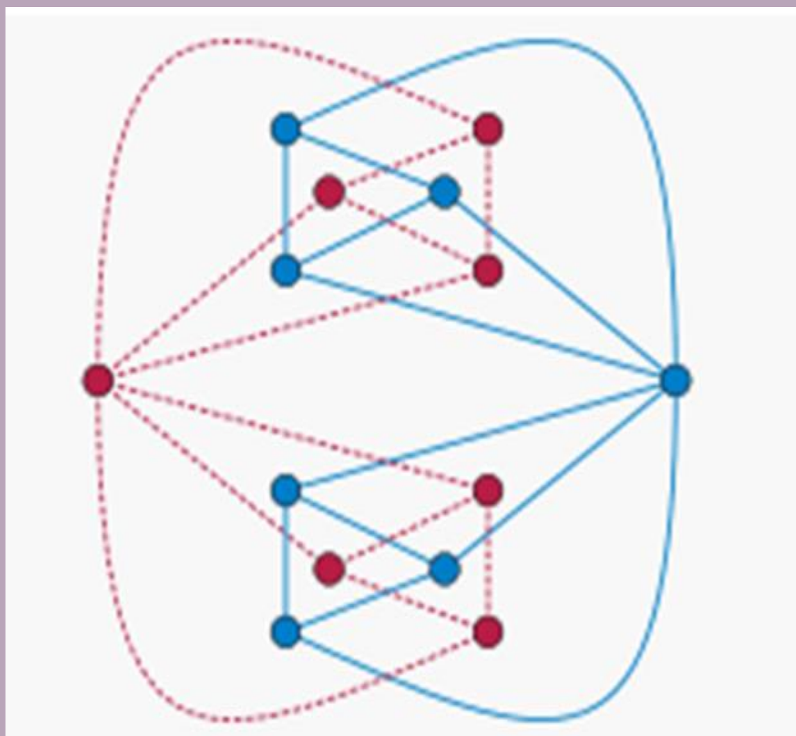
Two red graphs are duals for the blue one, but they are not isomorphic.



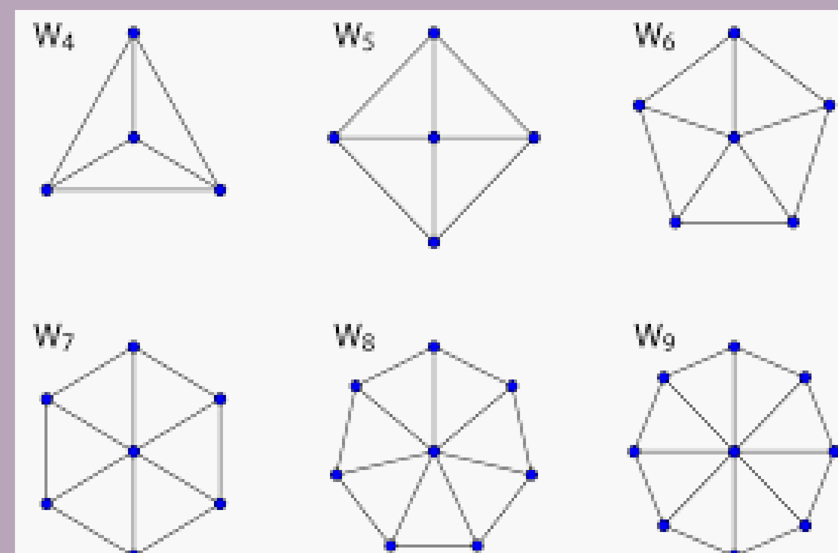
Theorem : Two planar graphs G_1 and G_2 are duals of each other iff there exists a one-to-one correspondence between their edge sets such that the circuits (cycles) in G_1 corresponds to cut-sets in G_2 and vice versa.

Self – dual graphs:

Definition: The self-dual graph is a graph that is dual to itself.

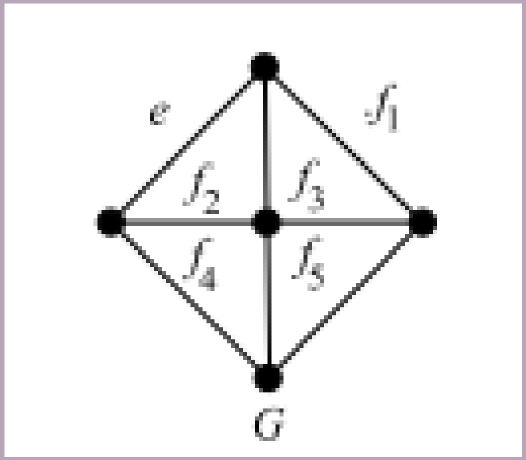


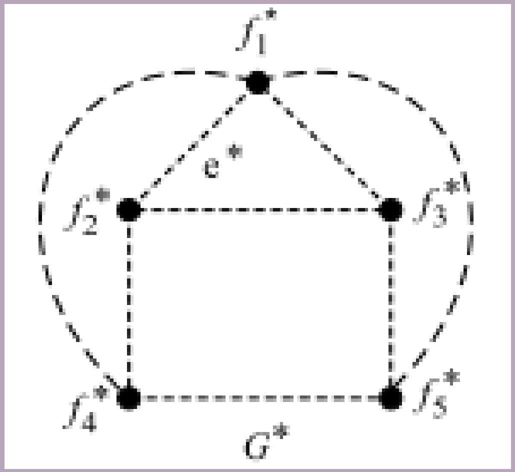
Wheel graphs are self- dual graphs.

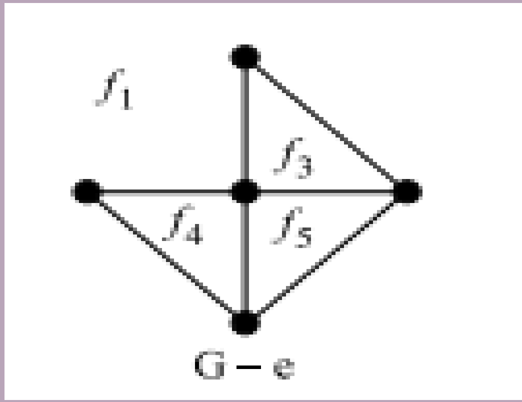


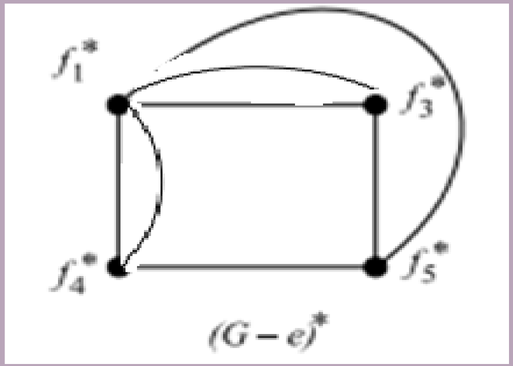
Dual of subgraph:

Let G be a planar graph and G^* be its dual. Let e be an edge in G, and the corresponding edge in G^* be e^*. Suppose that we delete edge e from G and then try to find the dual of $G - e$. If edge e was on the boundary of two regions, removal of e would merge these two regions into one. Thus, the dual $(G - e)^*$ can be obtained from G^* by deleting the corresponding edge e^* and then fusing the two end vertices of e^* in $G^* - e^*$. On the other hand, if edge e is not on the boundary, e^* forms a self-loop. In that case, $G^* - e^*$is the same as $(G - e)^*$. Thus, if a graph G has a dual G^*, the dual of any subgraph of G can be obtained by successive application of this procedure.


e
f_1
f_2
f_3
f_4
f_5
G


f_1^*
e^*
f_2^*
f_3^*
f_4^*
f_5^*
G^*

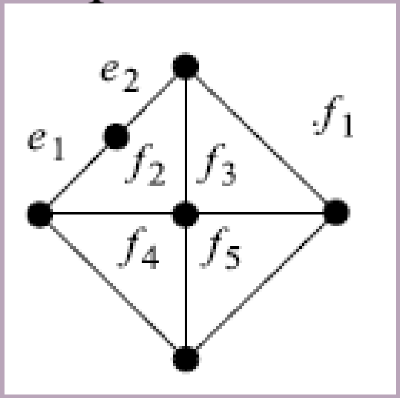

f_1
f_3
f_4
f_5
G − e


f_1^*
f_3^*
f_4^*
f_5^*
$(G-e)^*$

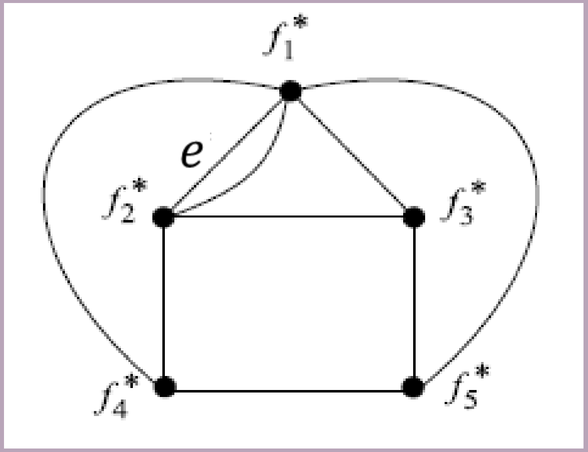
Dual of a Homeomorphic Graph:

Let G be a planar and G^* be its dual. Let e be an edge in G, and the corresponding edge in G^* be e^*. Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge e. It simply adds an edge parallel to e^* in G^*. Likewise, the reverse process of merging two edges in series will simply eliminate one of the corresponding parallel edges in G^*. Thus if a graph G has a dual G^*, the dual of any graph homeomorphic to G can be obtained from G^* by the above procedure.

G



G^*



Theorem : A graph has a dual if and only if it is planar.

Correspondences of Duals

We will now summarize a list of properties relating planar graphs and their duals.

Planar Graph G	Correspondence in G^*
An edge in G	An edge in G^*
A face in G	A vertex in G^*
A vertex in G	A face in G^*
A cycle of length k in G	A cutset in G^* with k edges
A cutset of G with k edges	A cycle of length k in G^*

THANK YOU

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