جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

Matrix Representations of Graphs

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Matrix Representations of Graphs

Lecture # 3

Graphs can be represented by using matrices. A matrix contains exactly the same information as a graph, but is more useful for computation and computer analysis.

There are several type of matrices, such as:

1) Incidence Matrix.

2) Cycle Matrix.

3) Cut-Set Matrix.

4) Adjacency Matrix.

5) Path Matrix.

Incidence Matrix of a Graph:

Definition: Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix defined by $A(G) = [a_{ij}]; 1 \leq i \leq n;\ 1 \leq j \leq m$, where

$$a_{ij} = \begin{cases} 1 & Where\ edge\ e_j\ incedent\ vertix\ v_i\ ; \\ 0 & Otherwise. \end{cases}$$

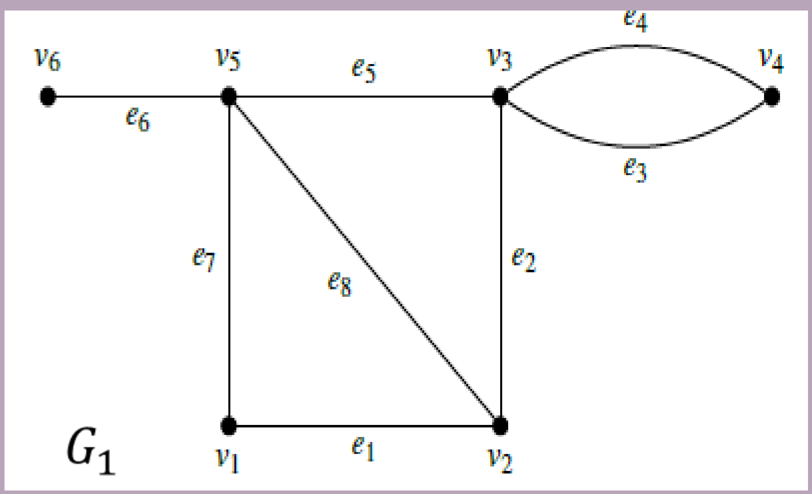
where the n rows of $A(G)$ correspond to the n vertices and the m columns of $A(G)$ correspond to m edges.

The incidence matrix contains only two types of elements, 0 and 1. Hence, this is clearly a binary matrix or a $(0,1)$ $-$matrix.

Simple connected Graph and Incidence Matrix

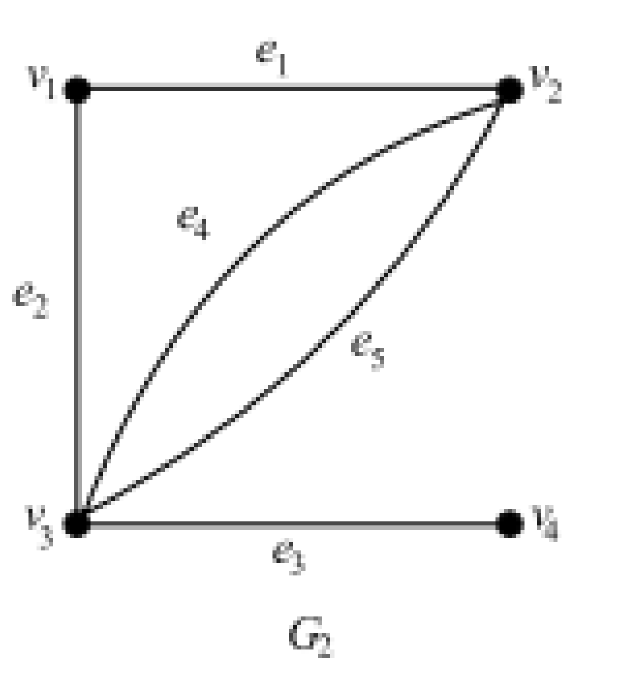
Example 1:

The incidence matrix of G_1 is



$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

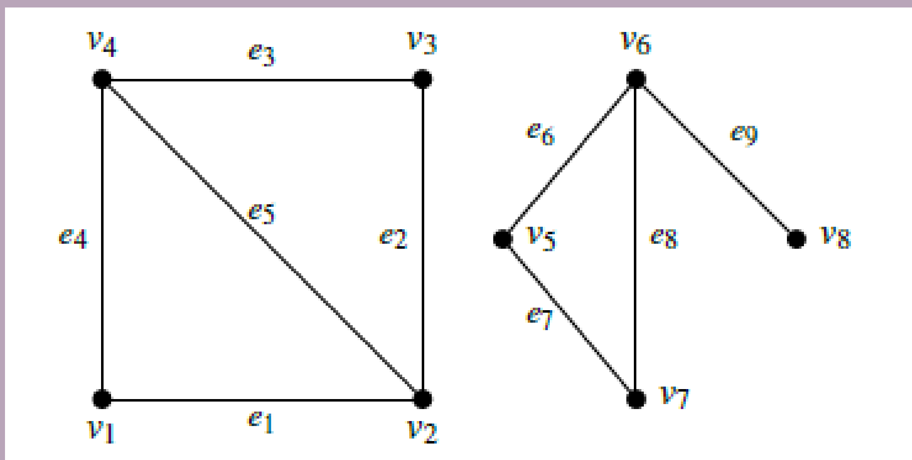
Example 2:



The incidence matrix of G_2 is

$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Simple disconnected Graph and Incidence Matrix Example:



A disconnected graph G with two components G_1 and G_2.

$$A(G) = \left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

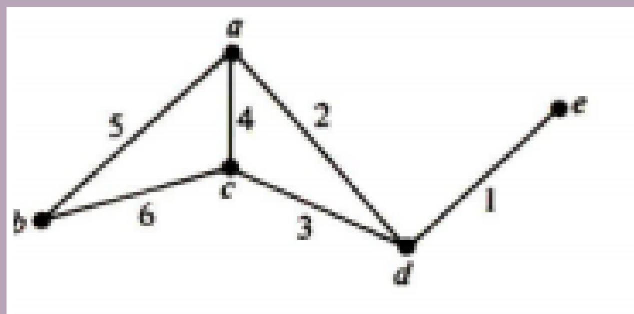
From the above examples, we have the following observations about the incidence matrix A of a graph G.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two ones.
2. The number of ones in each row equals the degree of the corresponding vertex.
3. A row with all zeros represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix.
5. If a graph G is disconnected and consists of two components G_1 and G_2, then its incidence matrix $A(G)$ can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

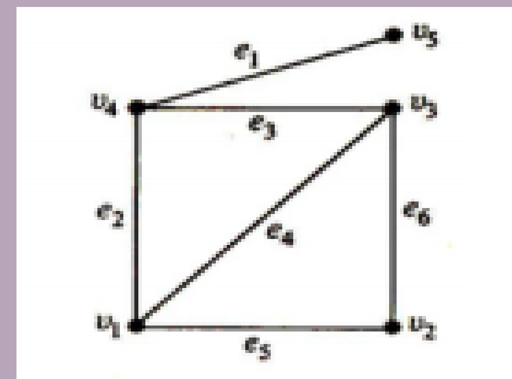
   where $A(G_1)$ and $A(G_2)$ are the incidence matrices of the components G_1 and G_2 of G. This observation results from the fact that no edge in G_1 is incident on vertices of G_2 and vice versa. Obviously, this is also true for a disconnected graph with any number of components.
6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabelling the vertices and edges of the same graph.

Theorem 1: Two graphs G_1 and G_2 are isomorphic iff their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.



G_1

G_2



- $$A(G_1) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $$A(G_2) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that the incidence matrix of G_1 and G_2 differ only by the permutation of column 5 and column 6.

Rank of Incidence Matrix:

Let G be a graph and let $A(G)$ be its incidence matrix. Now, each row in $A(G)$ is a vector over $GF(2)$ in the vector space of graph G. Let the row vectors be denoted by $A_1, A_2, \ldots, A_n$. Then,

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n. \end{bmatrix}$$

Since there are exactly two 1's in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus vectors $A_1, A_2, \ldots, A_n$ are linearly dependent. Therefore, $rankA < n$. Hence, $rankA \leq n-1$.

Theorem 2: If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G) = n - 1$.

Corollary : If G is a disconnected graph with k components, then rank $A(G) = n - k$.

Definition: Any $(n-1)\times m$ submatrix $A_f(G)$ of an $n\times m$ incidence matrix $A(G)$ of a connected graph G with no self- loop is called a reduced incidence matrix of the graph G. The vertex corresponding to the deleted row of $A(G)$ is called the reference vertex with respect to this reduced incidence matrix.

The following result gives the nature of the incidence matrix of a tree:

Theorem : The reduced incidence matrix of a tree is non-singular.

Let H be a subgraph of a graph G, and let $A(H)$ and $A(G)$ be the incidence matrices of H and G respectively. Clearly, $A(H)$ is a submatrix of $A(G)$, possibly with rows or columns permuted. We observe that there is a one-one correspondence between each $n \times k$ submatrix of $A(G)$ and a subgraph of G with k edges, where k is a positive integer, $k < m$ and n being the number of vertices in G.

Theorem : Let $A(G)$ be the incidence matrix of a connected graph G with n vertices. An $(n-1) \times (n-1)$ submatrix of $A(G)$ is non-singular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitutes a spanning tree in G.

THANK YOU

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