جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

Cycle Matrix / Part 1

م. د. امين شامان امين

University of Anbar / College of Science / Dept. of Applied Mathematics

Cycle Matrix

Lecture # 4

Part # 1

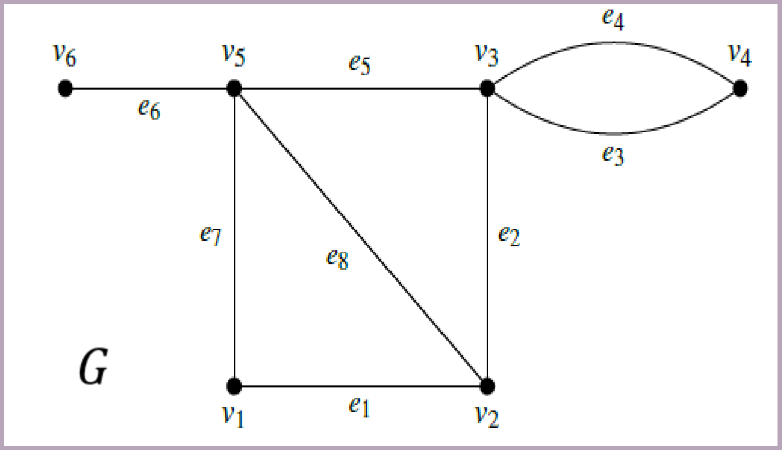
Recall a cycle or a circuit is a path that begins and ends on the same vertex.

Cycle Matrix:

Definition : Let G be a graph with e edges and q different cycles. The cycle matrix or circuit matrix of G, denoted by $B(G)$, is defined as a $(0,1)$ −matrix $B(G) = [b_{ij}]$ of order $q \times e$, such that

$$b_{ij} = \begin{cases} 1 & If\ the\ i-th\ cycle\ includes\ j-th\ edge; \\ 0 & Otherwise. \end{cases}$$

Example 1:



$$C_1 = \{e_1, e_7, e_8\}$$

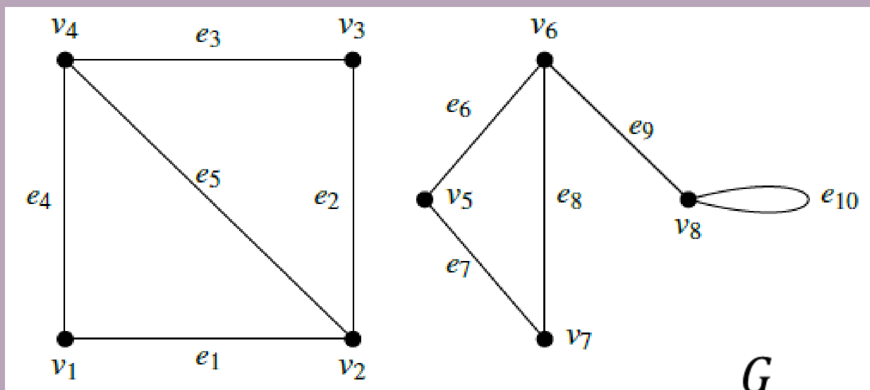
$$C_2 = \{e_2, e_5, e_8\}$$

$$C_3 = \{e_3, e_4\}$$

$$C_4 = \{e_1, e_2, e_5, e_7\}$$

$$B(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 2:



$$B(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_1 = \{e_1, e_4, e_5\}$$

$$C_2 = \{e_2, e_3, e_5\}$$

$$C_3 = \{e_1, e_2, e_3, e_4\}$$

$$C_4 = \{e_6, e_7, e_8\}$$

$$C_5 = \{e_{10}\}$$

$$B(G) = \begin{bmatrix} B(G_1) & 0 \\ 0 & B(G_2) \end{bmatrix}$$

In view of the above examples, we have the following observations regarding the cycle matrix $B(G)$ of a graph G:

1. A column of all zeros corresponds to a cut-edge. That is, an edge which does not belong to any cycle corresponds to an all-zero column in $B(G)$.
2. Each row of $B(G)$ is a cycle vector.
3. A cycle matrix has the property of representing a self-loop and the corresponding row has a single 1.
4. The number of 1's in a row is equal to the number of edges in the corresponding cycle.
5. If the graph G is separable (or disconnected) and consists of two blocks (or components) H_1 and H_2, then the cycle matrix $B(G)$ can be written in a block-diagonal form as

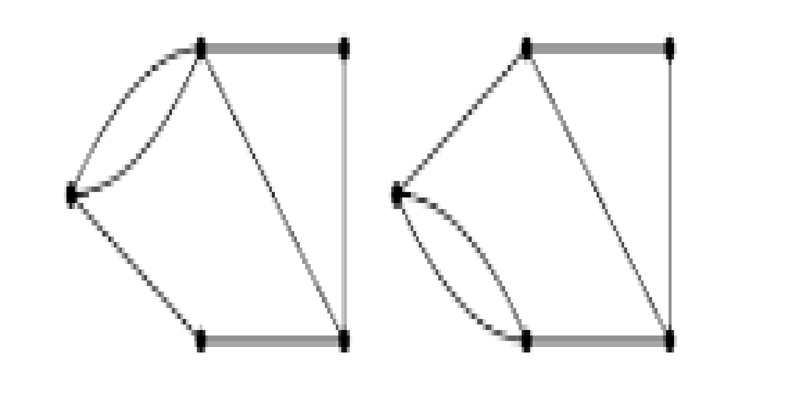
$$B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix}$$

where $B(H_1)$ and $B(H_2)$ are the cycle matrices of H_1 and H_2. This is obvious from the fact that cycles in H_1 have no edges belonging to H_1 and vice versa.

6. Permutation of any two rows or columns in a cycle matrix corresponds to relabeling the cycles and the edges.

7. We know two graphs G_1 and G_2 are 2-isomorphic if and only if they have cycle correspondence. Thus two graphs G_1 and G_2 have the same cycle matrix if and only if G_1 and G_2 are 2-isomorphic. This implies that the cycle matrix does not specify a graph completely, but only specifies the graph within 2-isomorphism.

For example, the two graphs given below have the same cycle matrix. They are 2-isomorphic, but are not isomorphic.

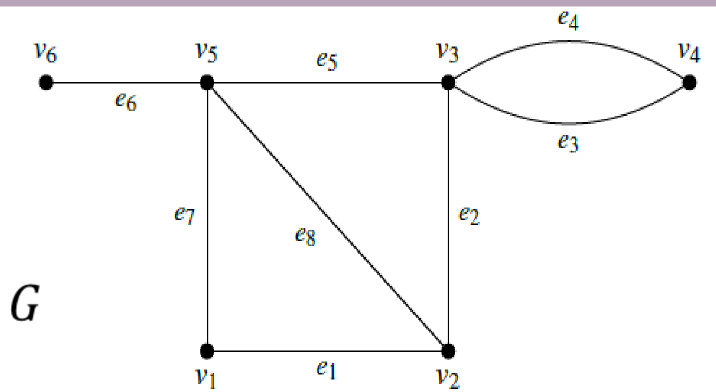


They have the same cycle matrix.

The cycle matrix of both graphs will be as follows:

$$B(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Theorem : If G is a graph without self-loops, with $A(G)$ and $\mathrm{B}(G)$ whose columns are arranged using the same order of edges, then every row of $\mathrm{B}(G)$ is orthogonal to every row of $A(G)$, that is $AB^T = BA^T = 0 \pmod 2$.



$$A(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv 0 \pmod 2.$$

THANK YOU

References:

1. S. Arumugam and S. Ramachandran, (2015), Invitation to graph theory, Scitech Publ., Kolkata, India.
2. G. S. Singh, (2013). Graph theory, Prentice Hall of India, New Delhi.
3. R. Balakrishnan and K. Ranganathan, (2012). A textbook of graph theory, Springer, New York.
4. J.A. Bondy and U.S.R Murty, (2008). Graph theory, Springer.
5. G. Agnarsson and R. Greenlaw, (2007). Graph theory: Modeling, applications & algorithms, Pearson Education, New Delhi.
6. G. Chartrand and P. Zhang, (2005). Introduction to graph theory, McGraw-Hill Inc.
7. G. Sethuraman, R. Balakrishnan, and R.J. Wilson, (2004). Graph theory and its applications, Narosa Pub. House, New Delhi.
8. D.B. West, (2001). Introduction to graph theory, Pearson Education Inc., Delhi.
9. V.K. Balakrishnan, (1997). Graph theory, McGrawhill, New York.
10. G. Chartrand and L. Lesniak, (1996). Graphs and digraphs, CRC Press.
11. J.A. Bondy and U.S.R Murty, (1976). Graph theory with applications, North-Holland, New York.