جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

Adjacency Matrix / Part 1

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Adjacency Matrix

Lecture # 7 Part 1

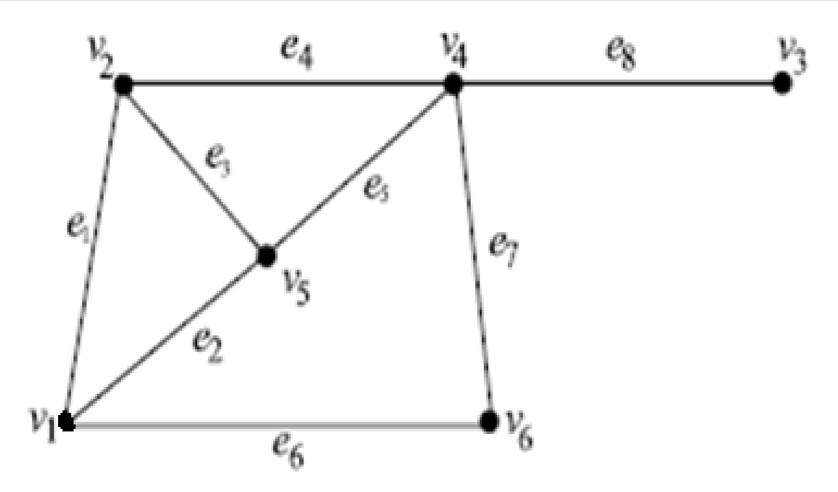
Recall for a graph $G = (V, E)$, two vertices $u, v \in V(G)$ are said to be adjacent if they are connected by an edge. That is $(u, v) \in E(G)$.

Adjacency Matrix

Definition: Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, $E = \{e_1, e_2, \dots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that:

$$x_{ij} = \begin{cases} 1 & \text{if there is an edge between } i\text{-th and } j\text{-th vertices,} \\ 0 & \text{if there is no edge between them.} \end{cases}$$

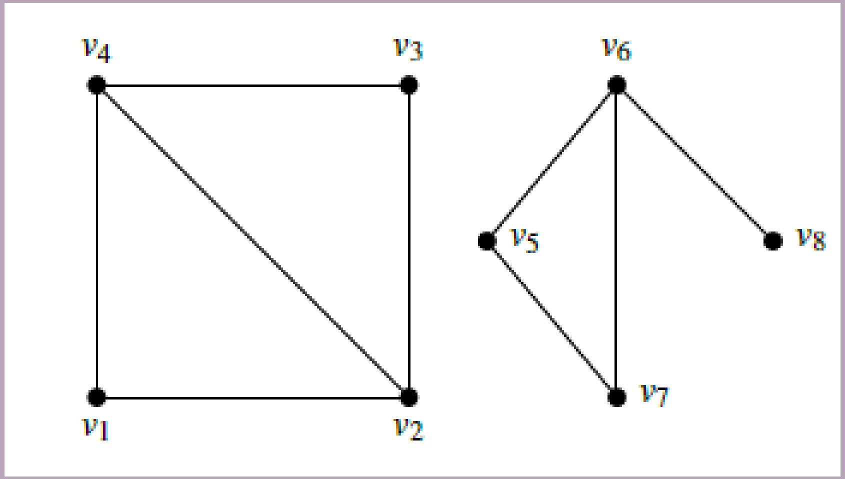
Example: Consider the graph G given in below



The adjacency matrix of G is given by:

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

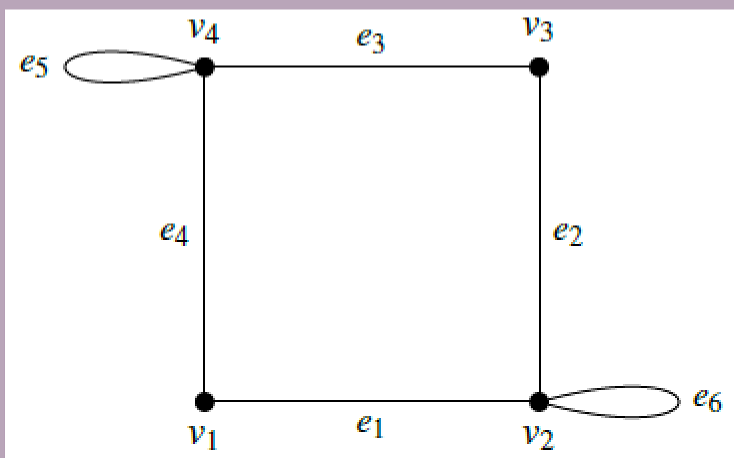
Example: Consider the following disconnected graph G with two components G_1 and G_2.



$$X(G) = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

The adjacency matrix of G is given by:

By the definition of the adjacency matrix of a graph G, we can notice that the adjacency matrices can be defined for the graphs having self-loops also. For example, consider the following graph and its adjacency matrix.



$$X(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Observations that can be made immediately about the adjacency matrix X of a graph G are:

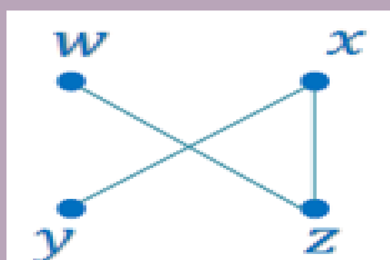
1. The entries along the principal diagonal of X are all $0's$ if and only if the graph has no self-loops. A self-loop at the ith vertex corresponds to $x_{ii} = 1$.

2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.

3. If the graph has no self-loops (and no parallel edges), the degree of a vertex equals the number of $1's$ in the corresponding row or column of X.

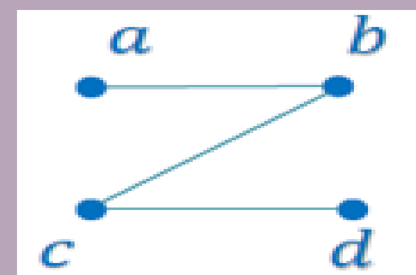
4. Permutations of rows and of the corresponding columns imply reordering the vertices. It must be noted, however, that the rows and columns must be arranged in the same order. Thus, if two rows are interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related : $X(G_2) = P^{-1}X(G_1)P$; where P is a permutation matrix.

G_1



$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

G_2



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{bmatrix}\overset{?}{=}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&0&0\\1&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$$

$$\overset{?}{=}\begin{bmatrix}0&0&0&1\\1&1&0&0\\0&0&1&1\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\overset{?}{=}\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{bmatrix}$$

$$X(G_2) = P^{-1}X(G_1)\,P \quad iff\ G_1 \cong G_2$$

Since $P^T = P^{-1}$

$$X(G_2) = P^T\,X(G_1)\,P \quad iff \quad G_1 \cong G_2$$

5. A graph G is disconnected and is in two components G_1 and G_2 if and only if its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \begin{bmatrix} X(G_1) & 0 \\ 0 & X(G_2) \end{bmatrix}$$

where $X(G_1)$ is the adjacency matrix of the component G_1 and $X(G_2)$ is that of the component G_2. This partitioning clearly implies that there exists no edge joining any vertex in subgraph G_1 to any vertex in subgraph G_2.

6. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel) such that Q is the adjacency matrix of G.

THANK YOU

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