جامعة الانبار

كلية العلوم

قسم الرياضيات التطبيقية

نظرية البيانات

<u>مناقشـــــة</u>

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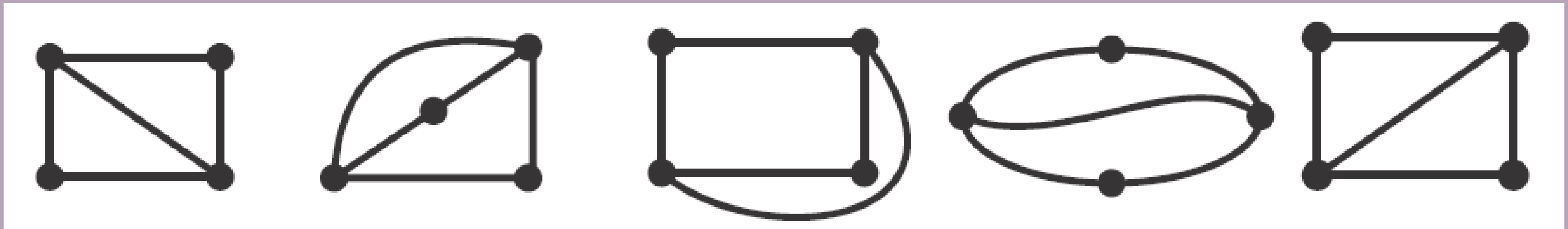
Isomorphic Graphs

Graph isomorphism is a phenomenon of existing the same graph in more than one form. Such graphs are called an isomorphic graphs .

In graph theory, an isomorphism of graphs G and H is a bijection between the vertex sets of G and H such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

Two graphs G_1 and G_2 are said to be isomorphic $(G_1 \cong G_2)$ if :

1. Their number of components (vertices & edges) is same.
2. Their edge connectivity is retained.



Necessary Conditions for Two Graphs to be Isomorphic:

For any two graphs G_1 & G_2 to be isomorphic, the following 4 conditions must be satisfied:

- $|V(G_1)| = |V(G_2)|$.
- $|E(G_1)| = |E(G_2)|$.
- Degree sequence of G_1 & G_2 are same.
- If the vertices $\{v_1, v_2, \dots, v_k\}$ form a cycle of length k in G_1, then the vertices $\{f(v_1), f(v_2), \dots, f(v_k)\}$ should also form a cycle of length k in the G_2.

Important Points: The previous 4 conditions are just the necessary conditions for any two graphs to be isomorphic.

- They are not at all sufficient to prove that the two graphs are isomorphic.
- If all the 4 conditions satisfy, even then it can’t be said that the graphs are surely isomorphic.
- However, if any condition violates, then it can be said that the graphs are surely not isomorphic.

Sufficient Conditions:

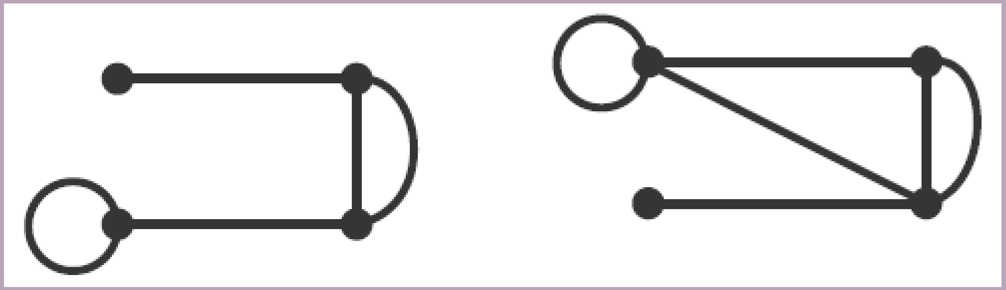
The following conditions are the sufficient conditions to prove that $G_1 \cong G_2$. If any one of these conditions satisfy, then it can be said that the graphs are surely isomorphic.

- $G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$, where G_1 and G_2 are simple graphs.
- $G_1 \cong G_2$ if their adjacency matrices are permuted equivalent. In other words, assume that $\mathrm{X}(G_1)$ & $\mathrm{X}(G_2)$ are the adjacency matrices of two isomorphic graphs G_1 & G_2, respectively, then there exist $v \times v$ permutation matrix P such that:

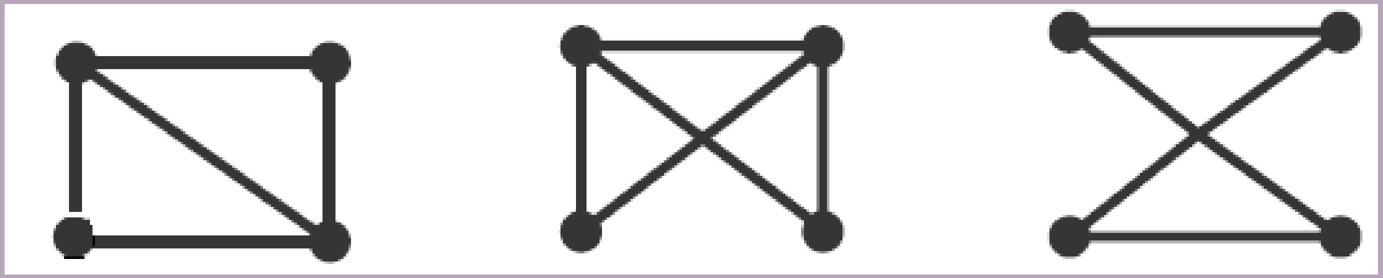
  $\mathrm{X}(G_1) = P^{-1}\,\mathrm{X}(G_2)P$.
- $G_1 \cong G_2$ iff their corresponding subgraphs (obtained by deleting some vertices in G_1 and their corresponding images in G_2) are isomorphic.

Example 1: Are the following two graphs isomorphic?

Solution: No, G_1 & G_2 are not isomorphic because G_1 has 5 edges and G_2 has 6 edges.



Example 2: Which of the following graphs are isomorphic?



G_1 G_2 G_3

Solution:

1) $|V(G_1)| = |V(G_2)| = |V(G_3)|$.

2) $|E(G_1)| = |E(G_2)| \neq |E(G_3)|$.

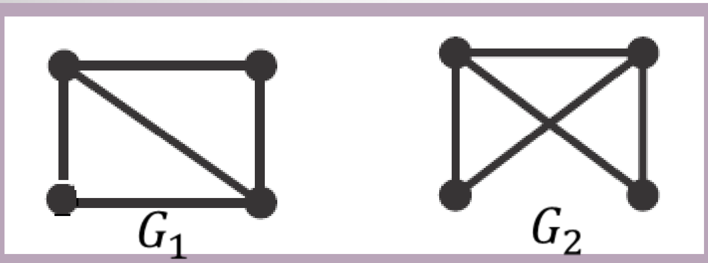
Then, G_3 neither isomorphic to G_1 nor isomorphic to G_2.

Now, let us check G_1 & G_2 .

3)

Degree sequence of G_1 is $\{2,2,3,3\}$.

Degree sequence of G_2 is $\{2,2,3,3\}$.



4) Cycles formed in G_1 are also formed in G_2.

Let us take the complement of G_1 and G_2.

$\overline{G_1}$ $\overline{G_2}$

Since , $\overline{G_1} \cong \overline{G_2}$

Thus, $G_1 \cong G_2$.

Example 3: Are the following two graphs isomorphic?

Solution:

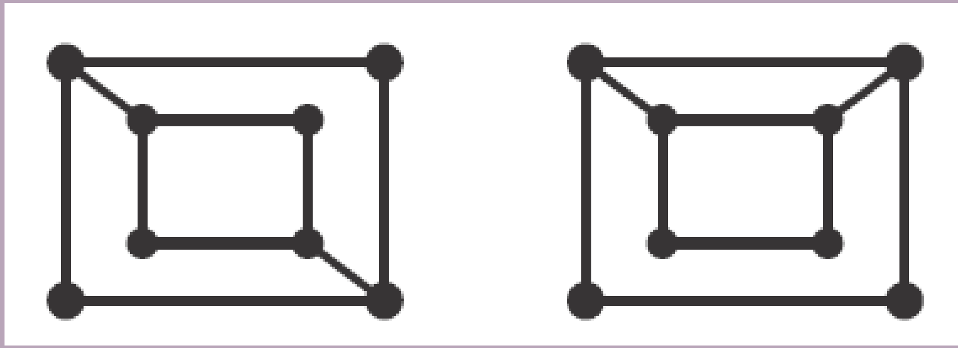
1) $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$.

2) Degree sequence of G_1 is $\{2,2,2,2,3,3,3,3\}$

Degree sequence of G_2 is $\{2,2,2,2,3,3,3,3\}$

3) In G_2, the vertices of degree 3 form 4- cycle but in G_1 the vertices of degree 3 does not form 4- cycle.

Thus, G_1 & G_2 are not isomorphic to each other.



G_1 G_2

Example 4: Find whether the following graphs are isomorphic.

Solution:

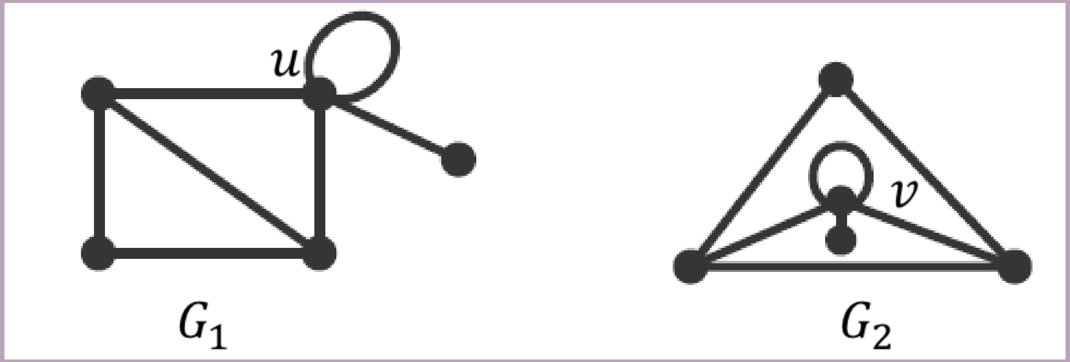
1) $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$.

2) Degree sequence of G_1 is $\{1,2,3,3,5\}$

   Degree sequence of G_2 is $\{1,2,3,3,5\}$

3) Cycle formed in G_1 are also formed in G_2.

Thus, all necessary conditions are satisfied for isomorphism.

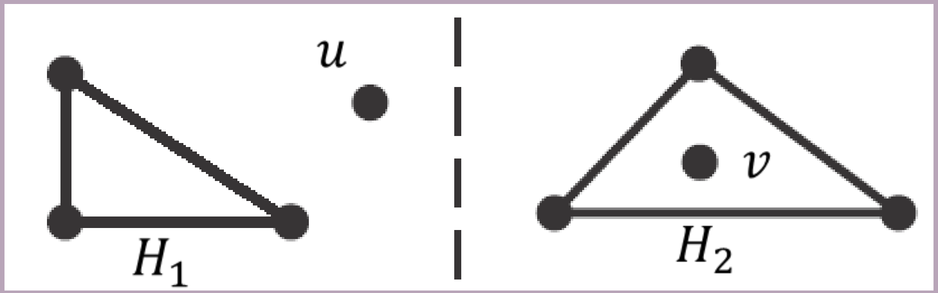


Deleting vertices u & v from corresponding graphs, we get:

Clearly, $H_1 \cong H_2$

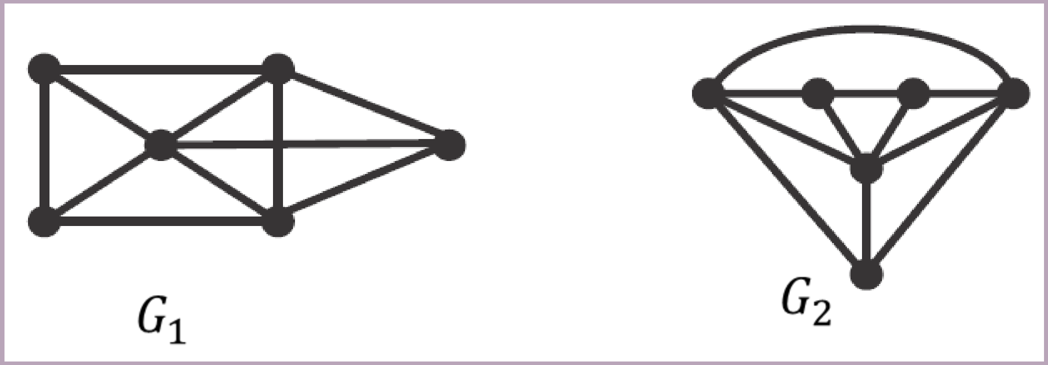
Then, $G_1 \cong G_2$.

Thus, G_1 & G_2 are isomorphic to each other.

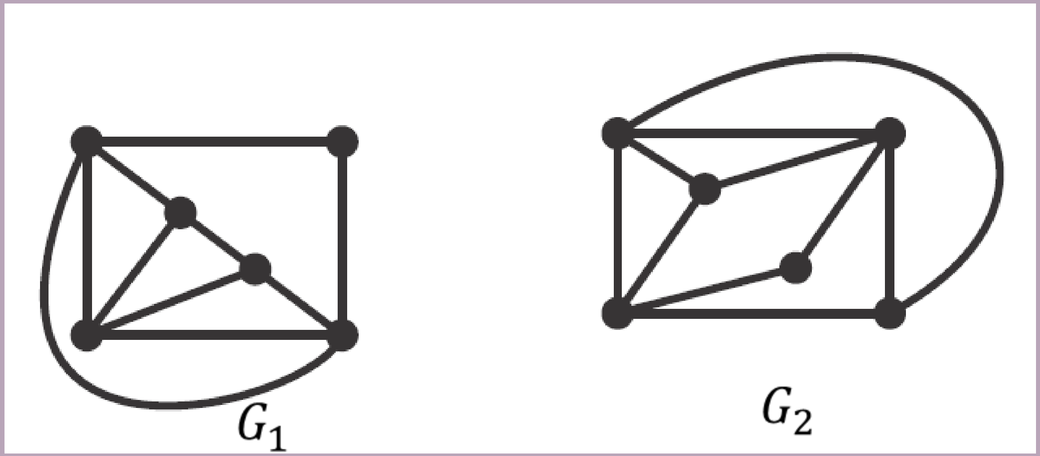


H.W: Find whether the following graphs are isomorphic.

1)



2)



THANK YOU

Special Matrix: Permutation Matrix

Definition: A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows. A permutation matrix called elementary if it is obtained by permutation of exactly two distinct rows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly, every permutation matrix has exactly one 1 in each row and column. It is easy to show that every an elementary permutation matrix is symmetric.

Note that an elementary permutation matrix corresponds to a transposition in S_n and every permutation matrix is a product of elementary matrices. In general, permutation matrix is not symmetric.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The set of all permutation matrices denoted by P_n and the $\#P_n = n!$. Indeed, there is one to one correspondence between P_n and S_n.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad n = 2 \to 2! = 2$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \qquad n = 3 \to 3! = 6$$

Since interchanging ith and jth rows of an identity is equivalent to interchanging its ith and jth columns, every *elementary* permutation matrix is symmetric, $P^\top = P$.

Since interchanging two rows is a self-reverse operation, every *elementary* permutation matrix is invertible and agrees with its inverse, $P = P^{-1}$ or $P^2 = I$.
A general permutation matrix does not agree with its inverse.

A product of permutation matrices is again a permutation matrix.
The inverse of a permutation matrix is again a permutation matrix. In fact, $P^{-1} = P^\top$.

Left multiplication by a permutation matrix rearranges the corresponding rows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix} = \begin{bmatrix} b & b & b \\ c & c & c \\ a & a & a \end{bmatrix}.$$

Right multiplication by a permutation matrix rearranges the corresponding columns:

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c & a & b \\ c & a & b \\ c & a & b \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f & d & e \\ i & g & h \\ c & a & b \end{bmatrix}.$$

Some power of a permutation matrix is the identity. For instance,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here $P^3 = I$ or $P^2 = P^{-1} = P^\top$.

THANK YOU

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Cut- Set Matrix
Fundamental Cut- Set Matrix
Rank of Matrix
Discussion # 1

Example:

$C_1 = \{e_1, e_2\}$

$C_2 = \{e_3, e_4\}$

$C_3 = \{e_4, e_5\}$

$C_4 = \{e_1, e_6\}$

$C_5 = \{e_2, e_6\}$

$C_6 = \{e_3, e_5\}$

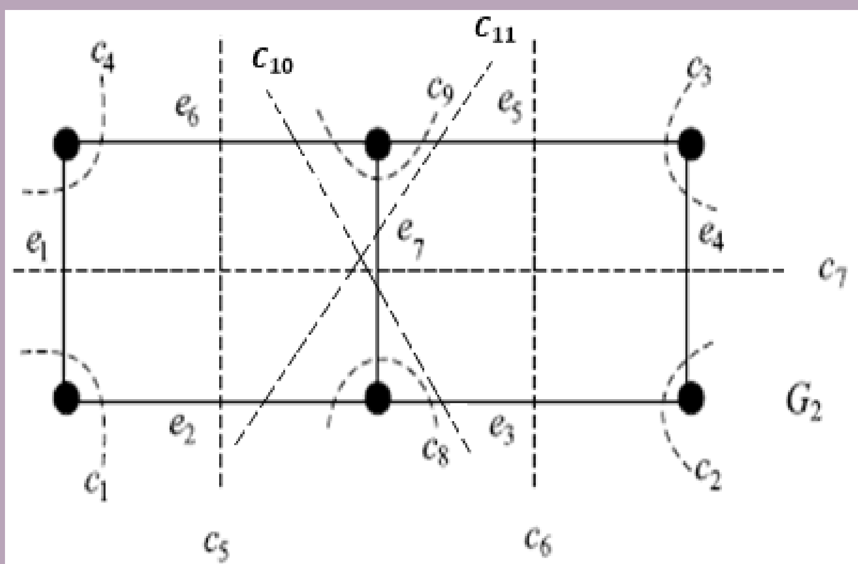
$C_7 = \{e_1, e_4, e_7\}$

$C_8 = \{e_2, e_3, e_7\}$

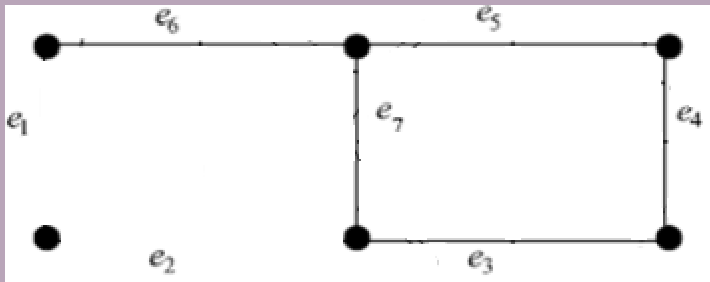
$C_9 = \{e_5, e_6, e_7\}$

$C_{10} = \{e_3, e_6, e_7\}$

$C_{11} = \{e_2, e_5, e_7\}$



$$
C(G_2) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \end{array}
\begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix} \end{array}
$$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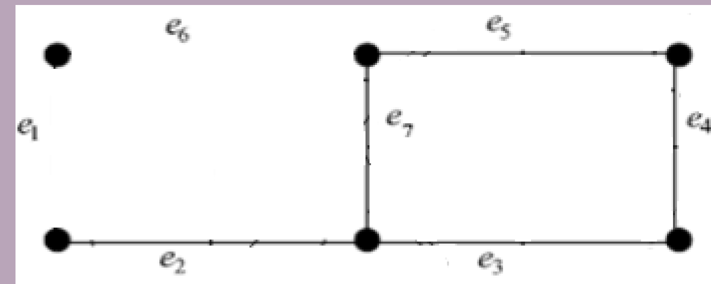
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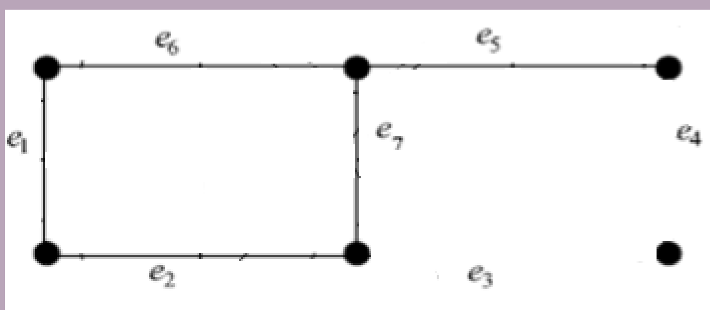
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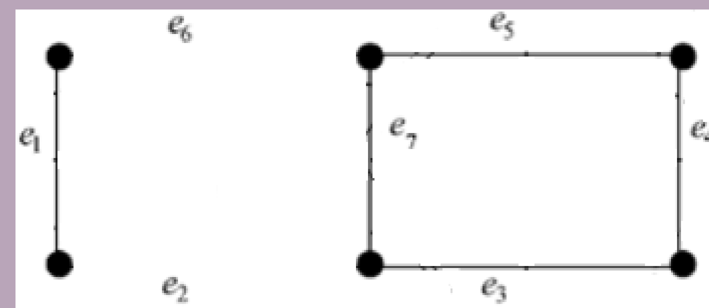
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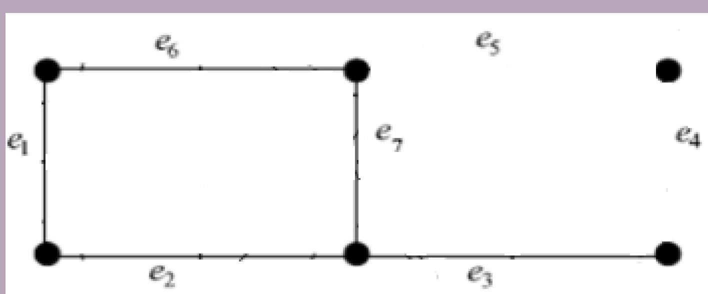
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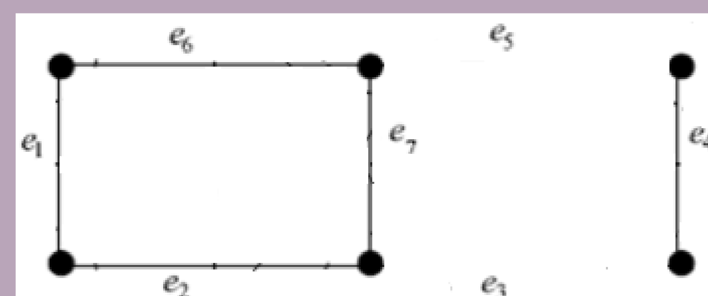
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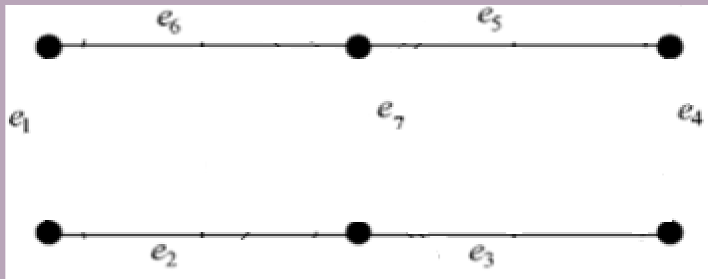
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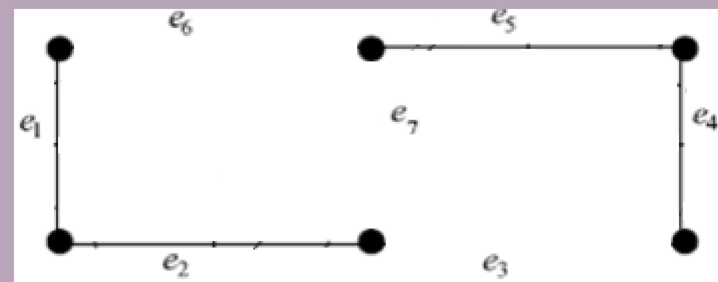
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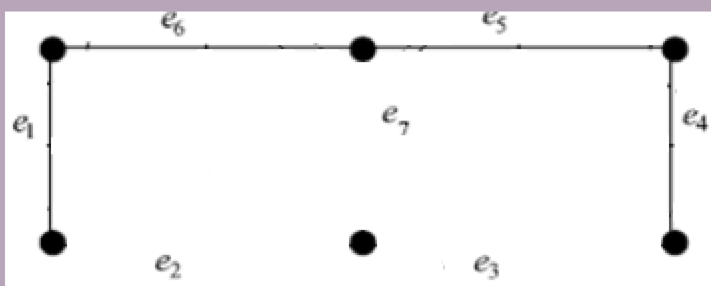
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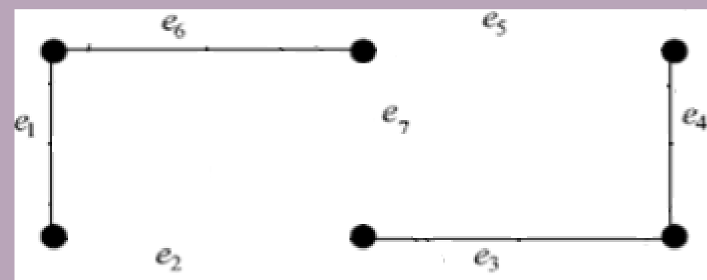
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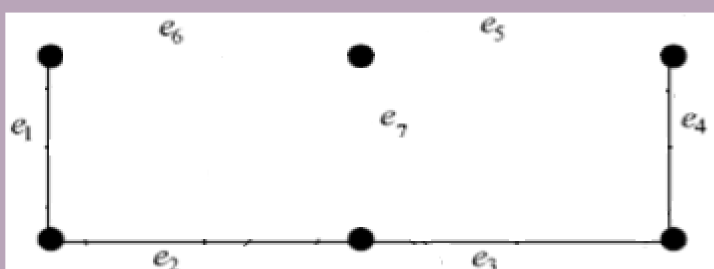
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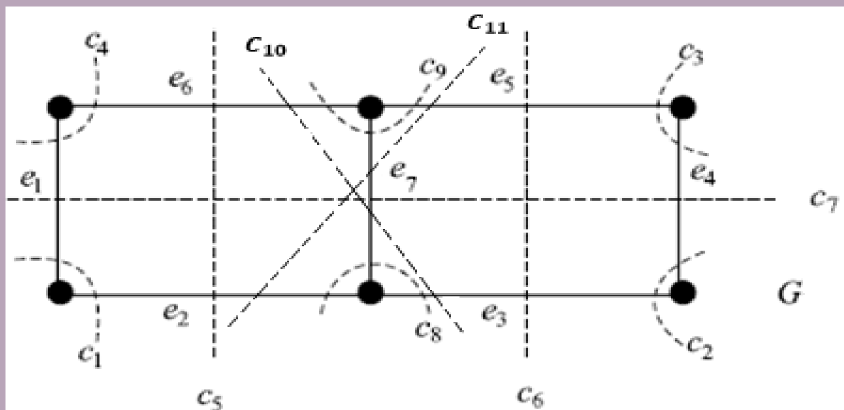
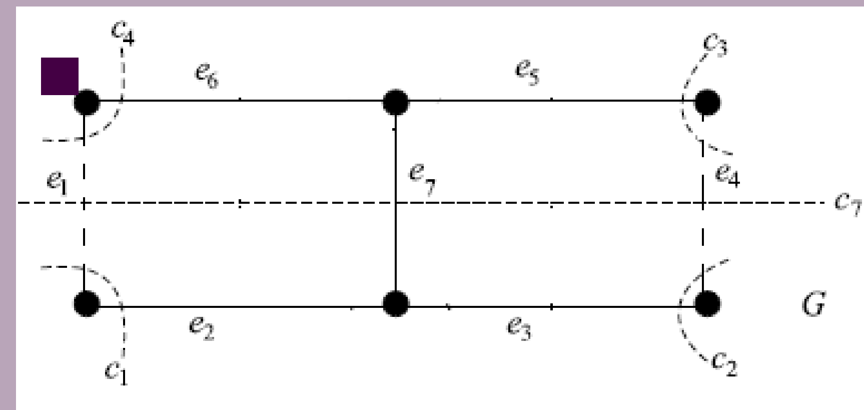
Fundamental Cut- Set:

Definition: Let T be a spanning tree of a connected graph G. A cut set S of G containing exactly one branch of T is called a fundamental cut-set of G with regard to T.





The cut- set of G as follows:

$C_1 = \{e_1, e_2\}$

$C_2 = \{e_3, e_4\}$

$C_3 = \{e_4, e_5\}$

$C_4 = \{e_1, e_6\}$

$C_5 = \{e_2, e_6\}$

$C_6 = \{e_3, e_5\}$

$C_7 = \{e_1, e_4, e_7\}$

$C_8 = \{e_2, e_3, e_7\}$

$C_9 = \{e_5, e_6, e_7\}$

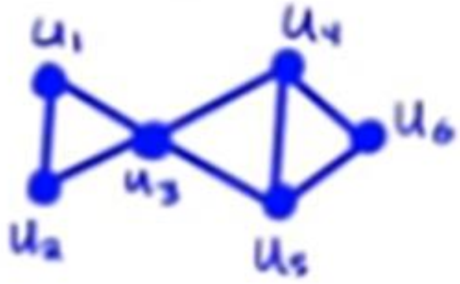
$C_{10} = \{e_3, e_6, e_7\}$

$C_{11} = \{e_2, e_5, e_7\}$

The fundamental cut- set of G as follows:

$C_1 = \{e_1, e_2\}$, $C_2 = \{e_3, e_4\}$,
$C_3 = \{e_4, e_5\}$, $C_4 = \{e_1, e_6\}$,
$C_7 = \{e_1, e_4, e_7\}$.

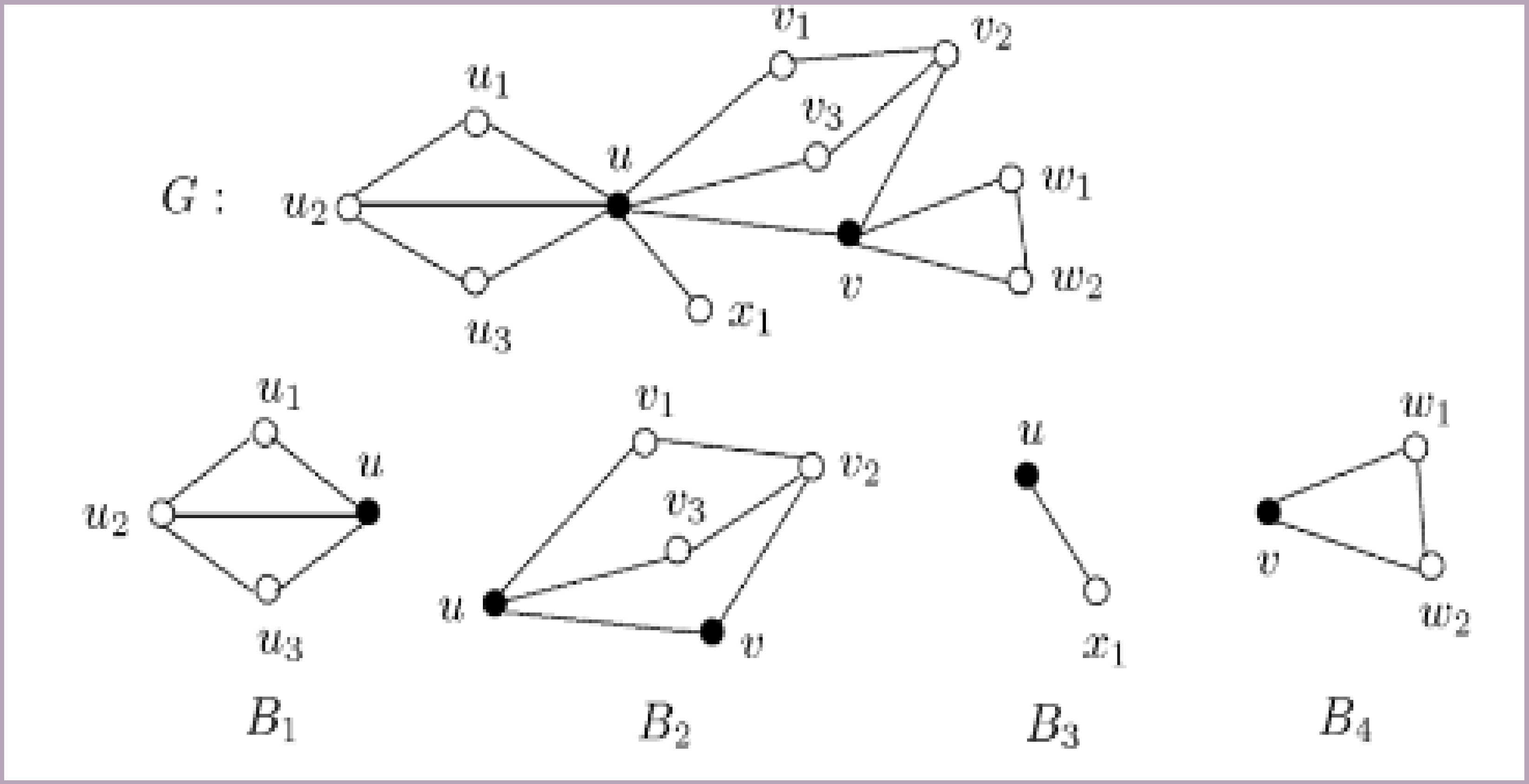
A <u>block</u> is a maximal nonseparable subgraph



Blocks:

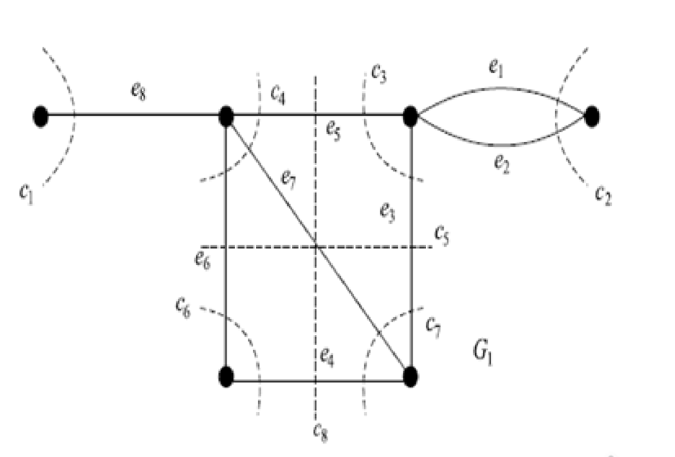
Notice that

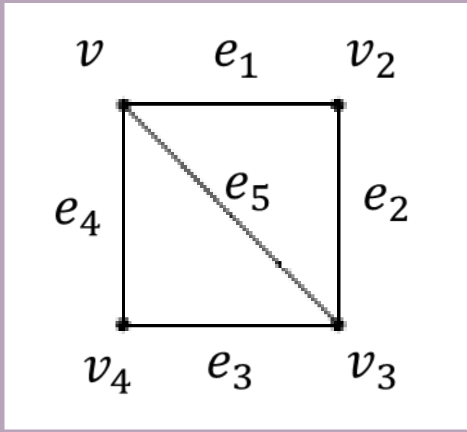
is nonseparable but not maximal nonseparable


G :
u_1
u_2
u_3
u
v_1
v_2
v_3
v
w_1
w_2
x_1
B_1
B_2
B_3
B_4

5. In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. That is, for a non-separable graph G, $C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. For example, in the graph G_1 (Example 1), the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of $C(G_1)$, left after deleting rows C_1, C_2, C_5, C_8 and columns e_1, e_2, e_8.

$$C(G_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$





non-separable graph

$C_1 = \{e_1, e_4, e_5\}$

$C_2 = \{e_1, e_2\}$

$C_3 = \{e_2, e_3, e_5\}$

$C_4 = \{e_3, e_4\}$

$C_5 = \{e_2, e_4, e_5\}$

$C_6 = \{e_1, e_3, e_5\}$

$$C(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$A(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$C(G)$ contains $A(G)$.

Rank of Matrix

Method 1 : By Normal form

Method 2 : By PAQ Form

Method 3 : By Echelon form

Method 4 : By Def^n of Rank

Method 1 : By Normal form

- Both Row & Col^m op^n are Allowed
- Reduce to Identity Matrix
- Normal form $\longrightarrow$ $A \sim \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$
- $\rho(A) = n$

Method 2 : PAQ form

- Both Row & Col^m op^n are Allowed
- $[A]_{m \times n} = I_m \, A \, I_n$
- $\rho(A)$ = Size of Identity Matrix

Method 3 : Echelon Form

- ONLY Row op^n are allowed
- Reduce Matrix to, Upper Triangular Matrix
- $\rho(A)$ = no. of NON-Zero Rows

Method 4 : By Def^n

"Rank of Matrix is defined as the order of largest Square Matrix whose Determinant is NOT ZERO

THANK YOU

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