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College: College of Science

Department: Physics

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Stage: Third stage

Subject: Optics 1

Lecture title: Fermat's Principle

1.5 Fermat's Principle

It was proposed by **Hero** long time ago what has since become known as a variational principle. In this treatment of reflection, it is assumed that the path taken by light in going from some point S to a point P via a reflecting surface was the shortest possible one. This can be seen rather easily in Fig. 1.13, which depicts a point source S emitting a number of rays that are then "reflected" toward P. Presumably, only one of these paths will have any physical reality. If we draw the rays as if they emanated from $\acute{S}$ (the image of S), none of the distances to P will have been altered (i.e., SAP = $\acute{S}AP$, SBP = $\acute{S}BP$, etc.). But obviously the straight-line path $\acute{S}BP$, which corresponds to θ_i= θ_t, is the shortest possible one.

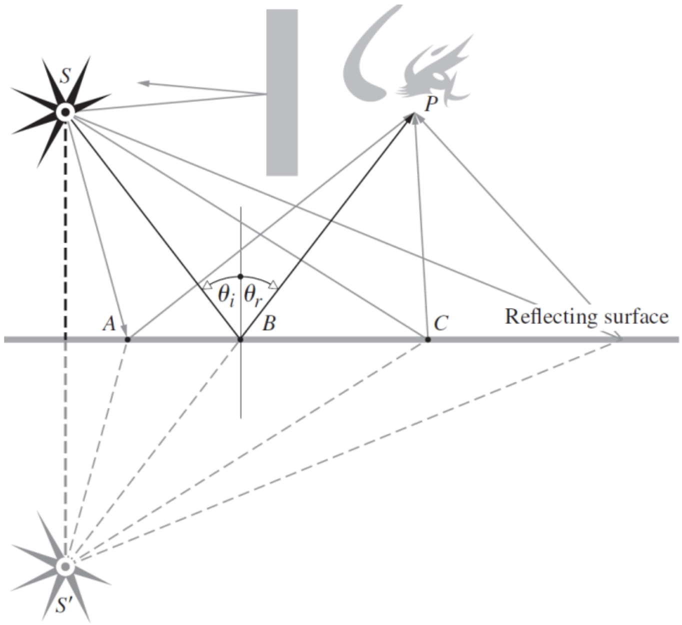

Figure 1. 1. Minimum path from the source S to the observer's eye at P.

Until in 1657 **Fermat** propounded his celebrated *Principle of Least Time*, which encompassed both reflection and refraction. A beam of light traversing an interface does not take a straight line or *minimum spatial path* between a point in the incident medium and one in the transmitting medium. Fermat consequently reformulated Hero's statement to read: *the actual path between two points taken by a beam of light is the one that is traversed in the least time*. As we shall see, even this form of the statement is incomplete and a bit erroneous at that. As an example of the application of the principle to the case of refraction, refer to Fig. 1.14, where we minimize t, the transit time from S to P, with respect to the variable x. In other words, changing x shifts point O, changing the ray from S to P. The smallest transit time will then presumably coincide with the actual path. Hence

$$t = \frac{\overline{SO}}{v_i} + \frac{\overline{OP}}{v_t}$$

Or

$$t = \frac{(h^2+x^2)^{1/2}}{v_i} + \frac{[b^2+(a-x)^2]^{1/2}}{v_t}$$

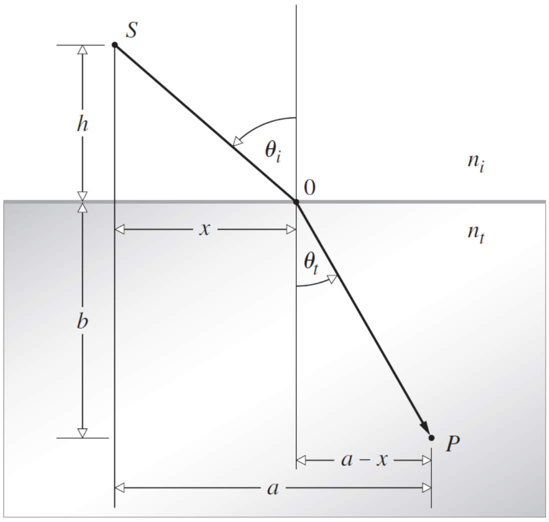

Figure 1. 2. Fermat's Principle applied to refraction.

To minimize *t(x)* with respect to variations in *x*, we set *dt>dx* = 0, that is,

$$\frac{dt}{dx} = \frac{x}{v_i(h^2 + x^2)^{1/2}} + \frac{-(a-x)}{v_t[b^2 + (a-x)^2]^{1/2}} = 0$$

Using the diagram, we can rewrite the expression as

$$\frac{sin\theta_i}{v_i} = \frac{sin\theta_t}{v_t}$$

which is no less than Snell's Law. If a beam of light is to advance from S to *P* in the least possible time, it must comply with the Law of Refraction. Suppose that we have a stratified material composed of m layers, each having a different index of refraction, as in Fig. 1.15. The transit time from *S* to *P* will then be

$$t = \frac{S_1}{v_1} + \frac{S_2}{v_2} + \cdots + \frac{S_m}{v_m}$$

$$t = \sum_{i=1}^{m} s_i/v_i$$

where S_i and v_i are the path length and speed, respectively, associated with the ith contribution. Thus

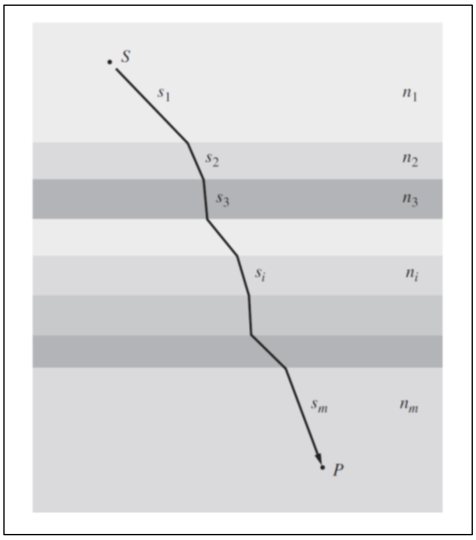

Figure 1. 3. A ray propagating through a layered material.

$$t = \frac{1}{c}\sum_{i=1}^{m} n_i\, s_i \qquad (1.6)$$

in which the summation is known as the **optical path length** (OPL) traversed by the ray, in contrast to the spatial path length $\sum_{i=1}^{m} s_i$. Clearly, for an inhomogeneous medium where n is a function of position, the summation must be changed to an integral:

$$OPL = \int_{S}^{P} n(s)ds \qquad (1.7)$$

The optical path length corresponds to the distance in vacuum equivalent to the distance traversed (s) in the medium of index n. That is, the two will correspond to the same number of wavelengths, $(OPL)/\lambda_o = s/\lambda$), and the same phase change as the light advances.
Inasmuch as $n = (OPL)/c$, we can restate Fermat's Principle: *light, in going from point S to P, traverses the route having the smallest optical path length*.

Fermat's Principle is not so much a computational device as it is a concise way of thinking about the propagation of light. It is a statement about the grand scheme of things without any concern for the contributing mechanisms, and as such it will yield insights under a myriad of circumstances.

1.6 The Electromagnetic Approach

We have studied reflection and refraction from the perspectives of Scattering Theory. Yet another and even more powerful approach is provided by Electromagnetic Theory. Unlike the previous techniques, which say nothing about the incident, reflected, and transmitted radiant flux densities (i.e., I_r, I_i, I_t, respectively), Electromagnetic Theory treats these within the framework of a far more complete description.

1.6.1 Waves at an Interface

Suppose that the incident monochromatic light wave is planar, so that it has the form

$$\vec{E}_i = \vec{E}_{oi} exp\left[i(\vec{k}_i \cdot \vec{r} - w_i t)\right] \qquad (1.8)$$

Or more simply,

$$\vec{E}_i = \vec{E}_{oi} cos\left(\vec{k}_i \cdot \vec{r} - w_i t\right) \qquad (1.9)$$

where the surfaces of constant phase are those for which $\vec{k}_i \cdot \vec{r}$= constant. Assume that $\vec{E}_{oi}$is constant in time; that is, the wave is linearly or plane polarized. Note that just as the origin in time, $t = 0$, is arbitrary, so too is the origin O in space, where $\vec{r} = 0$. Thus, making no assumptions about their directions, frequencies, wavelengths, phases, or amplitudes, we can write the reflected and transmitted waves as

$$\vec{E}_r = \vec{E}_{or} cos(\vec{k}_r \cdot \vec{r} - \omega_r t + \varepsilon_r) \tag{1.10}$$

$$\vec{E}_{it} = \vec{E}_{ot} cos(\vec{k}_t \cdot \vec{r} - \omega_t t + \varepsilon_t) \tag{1.11}$$

Here ε_r and ε_t are *phase constants* relative to $\vec{E}_i$and are introduced because the position of the origin is not unique. Figure 1.16 depicts the waves in the vicinity of the planar interface between two homogeneous lossless dielectric media of indices n_i and n_t.

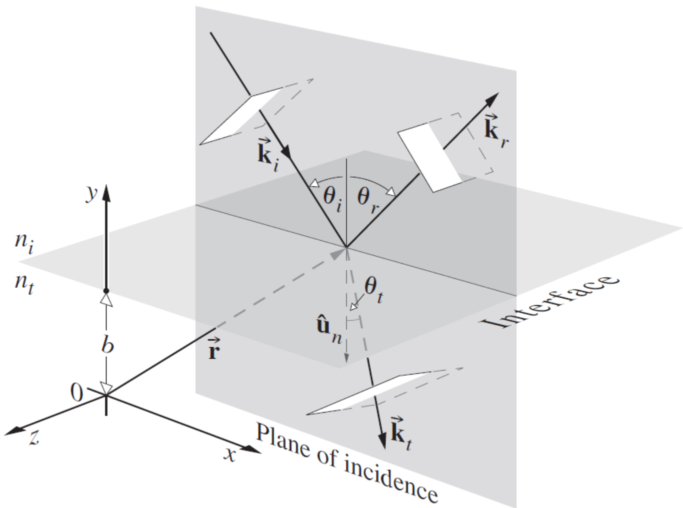

Figure 1. 4. Plane waves incident on the boundary between two homogeneous, isotropic, lossless dielectric media.

The laws of Electromagnetic Theory lead to certain requirements that must be met by the fields, and they are referred to as the boundary conditions. Specifically, one of these is that the component of the electric field $\vec{E}$ that is tangent to the interface must be continuous across it. An electromagnetic impinges from above on the interface, and the arrows represent either the incident and transmitted $\vec{E}$-fields or the corresponding $\vec{B}$-fields. For the moment we'll focus on the $\vec{E}$-fields. We draw a narrow closed (dashed) path C that runs parallel to the interface inside both media. Faraday's Induction Law

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\iint \frac{\partial \vec{\boldsymbol{B}}}{\partial t} \cdot d\vec{\boldsymbol{S}} \tag{1.12}$$

tells us that if we add up (via a line integral) the components of $\vec{E}$ parallel to the path elements $d\vec{\ell}$, each one times $d\vec{\ell}$, over the whole path C, the result (a voltage difference) will equal the time rate-of-change of the magnetic flux through the area bounded by C. But if we make the dashed loop very narrow there will be no flux through C, and the contribution to the line integral (moving right) along the top That way the net voltage drop around C will be zero. If the tangential components of $\vec{E}_i$ and $\vec{E}_t$ in the immediate vicinity of the interface are equal (e.g., both pointing to the right), because the paths reverse direction above and below the interface, the integral around C will indeed go to zero. In other words, the total tangential component of $\vec{E}$on one side of the surface must equal that on the other.

References:

1- Principles of optics- Max Born

2- Optics,-Eugene-Hecht

3- Optics and photonics an introduction