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Lecture title: The Fresnel Equations

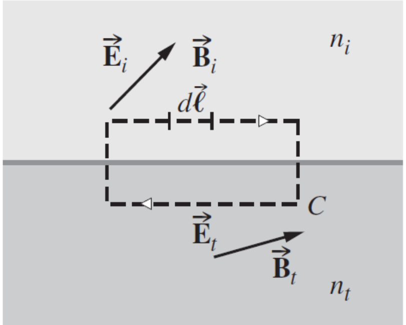

Figure 1. 1. Boundary conditions at the interface between two dielectrics.

Since $\hat{u}_n$ is the unit vector normal to the interface, regardless of the direction of the electric field within the wavefront, the cross-product of it with $\hat{u}_n$will be perpendicular to $\hat{u}_n$ and therefore tangent to the interface. Hence

$$\hat{u}_n \times \vec{E}_i + \hat{u}_n \times \vec{E}_r = \hat{u}_n \times \vec{E}_t \tag{1.13}$$

or $$\hat{u}_n \times \vec{E}_{0i} \cos\left(\vec{k}_i \cdot \vec{r} - \omega_i t\right) + \hat{u}_n \times \vec{E}_{0r} \cos\left(\vec{k}_r \cdot \vec{r} - \omega_r t + \varepsilon_r\right)$$

$$= \hat{u}_n \times \vec{E}_{0t} \cos\left(\vec{k}_t \cdot \vec{r} - \omega_t t + \varepsilon_t\right) \tag{1.14}$$

This relationship must obtain at any instant in time and at any point on the interface ($y = b$). Consequently, $\vec{E}_i$, $\vec{E}_r$, and $\vec{E}_t$ must have precisely the same functional dependence on the variables t and r, which means that

$$(\vec{k}_i \cdot \vec{r} - \omega_i t)|_{y=b} = (\vec{k}_r \cdot \vec{r} - \omega_r t + \varepsilon_r)|_{y=b}$$

$$= (\vec{k}_t \cdot \vec{r} - \omega_t t + \varepsilon_t)|_{y=b} \tag{1.15}$$

With this as the case, the cosines in Eq. (1.14) cancel, leaving an expression independent of t and r, as indeed it must be. Inasmuch as this has to be true for all values of time, the coefficients of t must be equal, to wit

$$\omega_i = \omega_r = \omega_t \tag{1.16}$$

Recall that the electrons within the media are undergoing (linear) forced vibrations at the frequency of the incident wave. Whatever light is scattered has that same frequency. Furthermore

$$(\vec{k}_i \cdot \vec{r})|_{y=b} = (\vec{k}_r \cdot \vec{r} + \varepsilon_r)|_{y=b} = (\vec{k}_t \cdot \vec{r} + \varepsilon_t)|_{y=b} \tag{1.17}$$

wherein $\vec{r}$ terminates on the interface. The values of ε_r and ε_t correspond to a given position of O, and thus they allow the relation to be valid regardless of that location. (For example, the origin might be chosen such that $\vec{r}$ was perpendicular to $\vec{k}_i$ but not to $\vec{k}_r$ or $\vec{k}_t$.) From the first two terms we obtain

$$[(\vec{k}_i - \vec{k}_r) \cdot \vec{r}]|_{y=b} = \varepsilon_r \qquad (1.18)$$

this expression simply says that the endpoint of $\vec{r}$ sweeps out a plane (which is of course the interface) perpendicular to the vector $(\vec{k}_i - \vec{k}_r)$. To phrase it slightly differently, $(\vec{k}_i - \vec{k}_r)$ is parallel to $\hat{u}_n$. Notice, however, that since the incident and reflected waves are in the same medium, $k_i = k_r$. From the fact that $\vec{k}_i - \vec{k}_r$ has no component in the plane of the interface, that is, $\hat{u}_n \times \vec{k}_i - \vec{k}_r = 0$, we conclude that

$$k_i sin\theta_i = k_r sin\theta_r$$

Hence, we have the law of reflection; that is

$$\theta_i=\theta_r$$

Furthermore, since $(\vec{k}_i - \vec{k}_r)$ is parallel to $\hat{u}_n$ all three vectors, $\vec{k}_i$, $\vec{k}_r$, and $\hat{u}_n$, are in the same plane, the plane-of-incidence. Again, from Eq. (1.17)

$$[(\vec{k}_i - \vec{k}_t) \cdot \vec{r}]|_{y=b} = \varepsilon_t \qquad (1.19)$$

And therefore $(\vec{k}_i - \vec{k}_r)$ is also normal to the surface. Thus $\vec{k}_i$, $\vec{k}_r$, $\vec{k}_t$, and $\hat{u}_n$ are all coplanar. As before, the tangential components of $\vec{k}_i$ and $\vec{k}_t$ must be equal, and consequently

$$k_i sin\theta_i = k_t sin\theta_t \qquad (1.20)$$

But because $\omega_i = \omega_t$, we can multiply both sides by c/ω_i to get

$$n_i sin\theta_i = n_t sin\theta_t$$

which is Snell's Law. Finally, if we had chosen the origin O to be in the interface, it is evident from Eqs. (1.18) and (1.19) that ε_r and ε_t would both have been zero. That arrangement, though not as instructive, is certainly simpler, and we'll use it from here on.

1.6.2 The Fresnel Equations

We have just found the relationship that exists among the phases of $\vec{E}_i(\vec{r},t)$, $\vec{E}_r(\vec{r},t)$, and $\vec{E}_t(\vec{r},t)$ at the boundary. There is still an interdependence shared by the amplitudes $\vec{E}_{0i}$, $\vec{E}_{0r}$, and $\vec{E}_{0t}$, which can now be evaluated. Suppose that a plane monochromatic wave is incident on the planar surface separating two isotropic media. Whatever the polarization of the wave, we shall resolve its $\vec{E}$- and $\vec{B}$- fields into components parallel and perpendicular to the plane-of-incidence and treat these constituents separately.

Case 1: **$\vec{E}$ perpendicular to the plane-of-incidence**. Assume that $\vec{E}$ is perpendicular to the plane-of-incidence and that $\vec{B}$ is parallel to it (Fig. 1.18). Recall that $E = vB$, so that

$$\hat{k} \times \vec{E} = v\vec{B} \qquad (1.21)$$

and

$$\hat{k} \times \vec{E} = 0 \qquad (1.22)$$

(i.e., $\vec{E}$, $\vec{B}$ and the unit propagation vector $\hat{k}$ form a right-handed system). Again, making use of the continuity of the tangential components of the $\vec{E}$- field, we have at the boundary at any time and any point

$$\vec{E}_{0i} + \vec{E}_{0r} = \vec{E}_{0t} \qquad (1.23)$$

where the cosines cancel. Realize that the field vectors as shown really ought to be envisioned at $y = 0$ (i.e., at the surface), from which they have been displaced for the sake of clarity. Note too that although $\vec{E}_r$ and $\vec{E}_t$ must be normal to the plane-of incidence by symmetry, we are guessing that they point outward at the interface when $\vec{E}_i$ does. The directions of the $\vec{B}$- fields then follow from Eq. (1.21). We will need to invoke another of the boundary conditions in order to get one more equation. The presence of material substances that become electrically polarized by the wave has a definite effect on the field configuration. Thus, although the tangential component of $\vec{E}$ (i.e., tangent to the interface) is continuous across the boundary, its normal component is not. Instead, the normal component of the product $\epsilon\vec{E}$ is the same on either side of the interface. Similarly, the normal component of $\vec{B}$ is continuous, as is the tangential component of $\mu^{-1}\vec{B}$. Here the magnetic effect of the two media appears via their permeabilities μ_i and μ_t. This boundary condition will be the simplest to use, particularly as applied to reflection from the surface of a conductor. Thus, the continuity of the tangential component of $\vec{B}/\mu$ requires that

$$-\frac{B_i}{\mu_i}\cos\theta_i + \frac{B_r}{\mu_i}\cos\theta_r = -\frac{B_t}{\mu_t}\cos\theta_t \qquad (1.24)$$

When the tangential component of the B- field points in the negative x-direction, as it does for the incident wave, it is entered with a minus sign. The left and right sides of the equation are the total magnitudes of $\vec{B}/\mu$ parallel to the interface in the incident and transmitting media, respectively. The positive direction is that of increasing x, so that the scalar components of B_i and B_t appear with minus signs. From Eq. (1.21) we have

$$B_i = E_i/v_i \qquad (1.25)$$

$$B_r = E_r/v_r \qquad (1.26)$$

$$B_t = E_t/v_t \qquad (1.27)$$

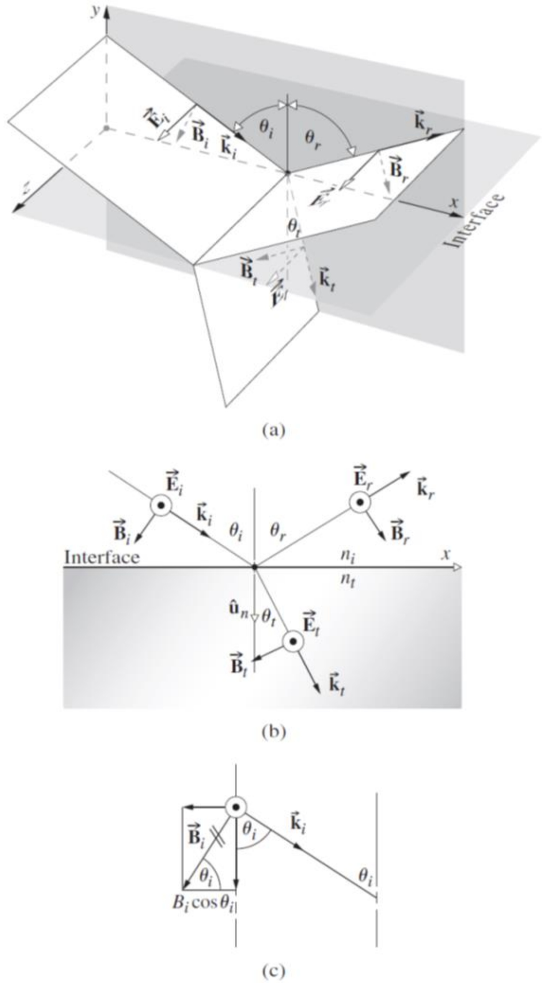

Figure 1. 2. An incoming wave whose $\vec{E}$- field is normal to the plane-of incidence. The fields shown are those at the interface; they have been displaced so the vectors could be drawn without confusion.

Since $v_i = v_r$ and $\theta_i = \theta_r$, Eq. (1.24) can be written as

$$\frac{1}{\mu_i v_i}(E_i - E_r)\cos\theta_i = \frac{1}{\mu_t v_t}E_t\cos\theta_t \qquad (1.28)$$

Making use of Eqs. (1.9), (1.10), and (1.11) and remembering that the cosines therein equal one another at y = 0, we obtain

$$\frac{n_i}{\mu_i}(E_{0i} - E_{0r})\cos\theta_i = \frac{n_t}{\mu_t}E_{0t}\cos\theta_t \tag{1.29}$$

Combined with Eq. (1.23)

$$\left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \tag{1.30}$$

and

$$\left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \tag{1.31}$$

The $\perp$ subscript serves as a reminder that we are dealing with the case in which $\vec{E}$ is perpendicular to the plane-of-incidence. These two expressions, *which are completely general statements applying to any linear, isotropic, homogeneous media*, are two of the **Fresnel Equations**. Most often one deals with dielectrics for which $\mu_i \approx \mu_t \approx \mu_0$; consequently, the common form of these equations is simply

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t} \tag{1.32}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2\,n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t} \tag{1.33}$$

Here $r_\perp$ denotes the **amplitude reflection coefficient**, and $t_\perp$ is the **amplitude transmission coefficient**.

References:

1- Principles of optics- Max Born

2- Optics,-Eugene-Hecht

3- Optics and photonics an introduction