3.5 Vector Product

(Given any two vectors $\vec{A}$ *and* $\vec{B}$, the vector product $(\vec{A} \times \vec{B})$ is defined as a third vector $\vec{C}$, which has a magnitude of $(AB\sin\theta)$).

$$\vec{C} = \vec{A} \times \vec{B} \qquad \text{Vector product}$$

$$C = AB\sin\theta \qquad \text{magnitude of vector product}$$

- The vector product$(\vec{A} \times \vec{B})$ is also called (**cross product**).

  Properties of the vector product:

1. It is not commutative $(\vec{A} \times \vec{B} = -\vec{B} \times \vec{A})$ Therefore, if you change the order of the vectors in a vector product, you must change the sign.
2. If $\vec{A}$ is parallel to $\vec{B}$ ($\theta = 0$ or $180°$), then

$$\vec{A} \times \vec{B} = 0 \quad \text{and} \quad \vec{A} \times \vec{A} = 0$$

3. If $\vec{A}$ is perpendicular to $\vec{B}$ ($\theta = 90°$), then

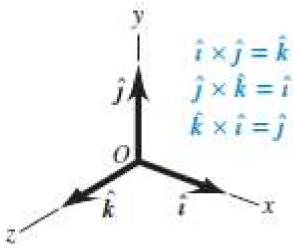
$$|\vec{A} \times \vec{B}| = AB$$

4. The vector product obeys the distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

5. The derivative of the vector product with respect to some variable such as (t) is: $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

- The cross products of the unit vectors ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$) obey the following rules: $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$
- $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$
- $\hat{k} \times \hat{\imath} = \hat{\jmath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$



The cross product of any two vectors $\vec{A}$ *and* $\vec{B}$ can be expressed in the following determinant form:

$$\vec{A}\times\vec{B}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}=\hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}+\hat{\jmath}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix}+\hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$=\hat{\imath}\,(A_yB_z-A_zB_y)+\hat{\jmath}\,(A_zB_x-A_xB_z)+\hat{k}\,(A_xB_y-A_yB_x)$$

Example (3.3):

Two vectors lying in the (*xy* plane) are given by the equations:

$$\vec{A}=2\hat{\imath}+3\hat{\jmath} \text{ and } \vec{B}=-\hat{\imath}+2\hat{\jmath}$$

Find $\vec{A}\times\vec{B}$ and verify that ($\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$).

Solution:

$\vec{A}\times\vec{B}=(2\hat{\imath}+3\hat{\jmath})\times(-\hat{\imath}+2\hat{\jmath})$

$=2\hat{\imath}\times(-\hat{\imath})+2\hat{\imath}\times 2\hat{\jmath}+3\hat{\jmath}\times(-\hat{\imath})+3\hat{\jmath}\times 2\hat{\jmath}$

$=0+4\hat{k}+3\hat{k}+0=7\hat{k}$

To verify that $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$:

$\vec{B}\times\vec{A}=(-\hat{\imath}+2\hat{\jmath})\times(2\hat{\imath}+3\hat{\jmath})$

$=(-\hat{\imath})\times 2\hat{\imath}+(-\hat{\imath})\times 3\hat{\jmath}+2\hat{\jmath}\times 2\hat{\imath}+2\hat{\jmath}\times 3\hat{\jmath}$

$=0-3\hat{k}-4\hat{k}+0=-7\hat{k}$

Therefore $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

Example (3.4):

Vector $\vec{A}$ has a magnitude of 6 units and it is in the direction of positive *x*- axis. Vector $\vec{B}$ has a magnitude of 4 units and lies in *xy*-plane making an angle 30° with *x*- axis. Find $\vec{A}\times\vec{B}$?

Solution:

$\vec{A}=6\,\hat{\imath}+0\,\hat{\jmath}+0\,\hat{k}$

$\vec{B}=4\,\hat{\imath}\cos 30+4\,\hat{\jmath}\sin 30+0\,\hat{k}=2\sqrt{3}\,\hat{\imath}+2\,\hat{\jmath}$

$$\vec{C}=\vec{A}\times\vec{B}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 0 \\ 2\sqrt{3} & 2 & 0 \end{vmatrix}=12\,\hat{k}$$

Chapter 4

(Motion in Two Dimensions)

4.1 The Position, Velocity, and Acceleration Vectors

We begin by describing the position of the particle by its **position vector** ($\vec{\mathbf{r}}$), drawn from the origin of some coordinate system to the location of the particle in the (xy plane) as shown in the figure.

At time t_i, the particle is at point (A), described by position vector $\vec{\mathbf{r}}_i$.

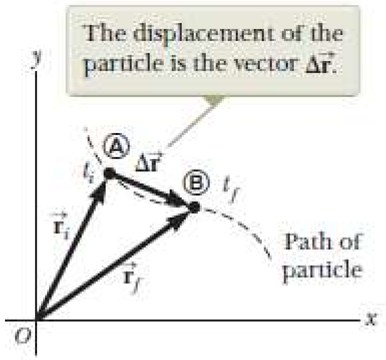
At some later time t_f , it is at point (B), described by position vector $\vec{\mathbf{r}}_f$.

The path from (A) to (B) is not necessarily a straight line. As the particle moves from (A) to (B) in the time interval ($\Delta t = t_f - t_i$), its position vector changes from $\vec{\mathbf{r}}_i$ to $\vec{\mathbf{r}}_f$.

We now define the **displacement vector (**$\Delta\vec{\mathbf{r}}$) for a particle as being (the difference between its final position vector and its initial position vector):

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_f \qquad \textbf{Displacement vector} \qquad (4.1)$$

As we see from the figure, the magnitude of ($\Delta\vec{\mathbf{r}}$) is *less* than the distance traveled along the curved path followed by the particle.



- The **average velocity** ($\vec{\mathbf{v}}_{ave}$) of a particle during the time interval (Δt) as the displacement of the particle divided by the time interval:

$$\vec{\mathbf{v}}_{ave} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} \qquad \textbf{Average velocity} \qquad (4.2)$$

- Multiplying or dividing a vector quantity by a positive scalar quantity such as (Δt) changes only the magnitude of the vector, not its direction. Because displacement is a vector quantity and the time interval is a positive scalar quantity, we conclude that the average velocity is a vector quantity directed along ($\Delta\vec{\mathbf{r}}$).

- The average velocity between points is independent of the path taken.
- The **instantaneous velocity** ($\vec{\mathbf{v}}$) is defined as (the limit of the average velocity $\frac{\Delta\vec{\mathbf{r}}}{\Delta t}$ as Δt approaches zero):

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt} \qquad \textbf{Instantaneous velocity} \qquad (4.3)$$

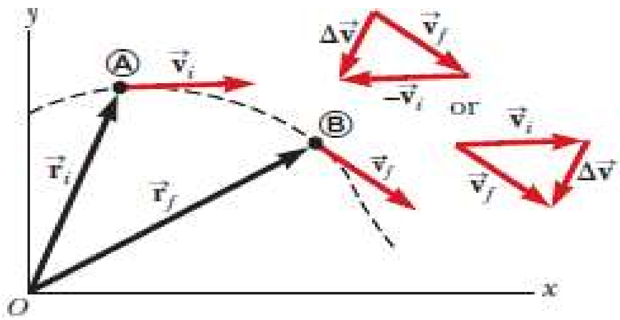
- The magnitude of the instantaneous velocity vector ($v = |\vec{\mathbf{v}}|$) of a particle is called the **speed** of the particle, which is a scalar quantity.
- The **average acceleration** ($\vec{\mathrm{a}}_{ave}$) of a particle is defined as (the change in its instantaneous velocity vector **(**$\Delta\vec{\mathbf{v}}$) divided by the time interval Δt during which that change occurs):

$$\vec{\mathrm{a}}_{ave} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i}{t_f - t_i} \qquad \textbf{Average acceleration} \qquad (4.4)$$

Average acceleration is a vector quantity.

- The **instantaneous acceleration** ($\vec{\mathrm{a}}$) is defined as (the limiting value of the ratio $\frac{\Delta\vec{\mathbf{v}}}{\Delta t}$ as Δt approaches zero):

$$\vec{\mathrm{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt} \qquad \textbf{Instantaneous acceleration} \qquad (4.5)$$



4.2 Two-Dimensional Motion with Constant Acceleration

Motion in two dimensions can be modeled as two *independent* motions in each of the two perpendicular directions associated with the x and y axes. That is, any influence in the y direction does not affect the motion in the x direction and vice versa.

The position vector for a particle moving in the xy plane can be written:

$$\vec{\mathbf{r}} = \hat{\imath}x + \hat{\jmath}y \qquad (4.6)$$

The velocity of the particle:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt}$$

$$\vec{\mathbf{v}} = \hat{\imath}\, v_x + \hat{\jmath}\, v_y \qquad (4.7)$$

To determine the final velocity at any time t, we obtain:

$$\vec{\mathbf{v}}_f = (v_{ix} + a_x t)\,\hat{\imath} + (v_{iy} + a_y t)\,\hat{\jmath} = (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath}) + (a_x\,\hat{\imath} + a_y\,\hat{\jmath})\, t$$

$\boxed{\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}\, t}$ Velocity vector as a function of time (4.8)

Similarly,

$$x_f = x_i + v_{ix}\, t + \frac{1}{2} a_x t^2 \quad \text{and} \quad y_f = y_i + v_{iy}\, t + \frac{1}{2} a_y t^2$$

Substituting these expressions into equation (4.6) (and labeling the final position vector ($\vec{\mathbf{r}}_f$) gives:

$$\vec{\mathbf{r}}_f = (x_i + v_{ix}\, t + \frac{1}{2} a_x t^2)\,\hat{\imath} + (y_i + v_{iy}\, t + \frac{1}{2} a_y t^2)\,\hat{\jmath}$$

$$= (x_i\,\hat{\imath} + y_i\,\hat{\jmath}) + (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath})\, t + \frac{1}{2}(a_x\,\hat{\imath} + a_y\,\hat{\jmath})\, t^2$$

$\boxed{\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}\, t^2}$ Position vector as a function of time (4.9)

Example (4.1):

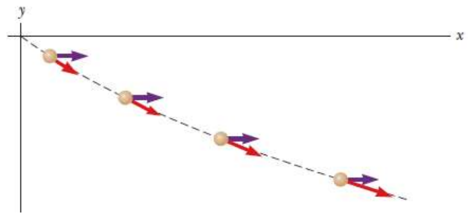
A particle moves in the xy plane, starting from the origin at ($t = 0$) with an initial velocity having an x - component of (20 m/s) and y- component of (-15 m/s). The particle experiences an acceleration in the x - direction, given by ($a_x = 4.0$ m/s^2).

(A) Determine the total velocity vector at any time.

(B) Calculate the velocity and speed of the particle at (t = 5.0 s) and the angle the velocity vector makes with the x- axis.

(C) Determine the x and y coordinates of the particle at any time t and its position vector at this time.

Solution:



(A) The components of the initial velocity tell us that the particle starts by moving toward the right and downward.

The x- component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y - component of velocity never changes from its initial value of (-15 m/s).

$\vec{v}_f = \vec{v}_i + \vec{a}\,t = (v_{ix} + a_x\,t)\,\hat{\imath} + (v_{iy} + a_y\,t)\,\hat{\jmath}$

$\vec{v}_f = [20 + 4\,t]\,\hat{\imath} + [-15 + 0\,t\,]\,\hat{\jmath}$

$\vec{v}_f = [(20 + 4\,t)\,\hat{\imath} - 15\,\hat{\jmath}\,]$ m/s

(B) $\vec{v}_f = [(20 + 4\,t)\,\hat{\imath} - 15\,\hat{\jmath}\,] = [\,\{20 + 4(5)\}\,\hat{\imath} - 15\,\hat{\jmath}] = (40\,\hat{\imath} - 15\,\hat{\jmath})$ m/s

The angle θ: $\theta = \tan^{-1}\frac{v_{yf}}{v_{xf}} = \tan^{-1}\frac{-15}{40} = -21°$

The negative sign for the angle θ indicates that the velocity vector is directed at an angle of 21° below the positive x - axis.

The speed of the particle as the magnitude of $\vec{v}_f$:

$$v_f = |\vec{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ v_f = 43 \text{ m/s}$$

(C) $x_f = v_{xi}t + \frac{1}{2}a_x t^2$

$x_f = (20\,t + 2\,t^2)$ m

$y_f = v_{yi}t = (-15\,t)$ m

The position vector of the particle at any time t :

$$\vec{\mathbf{r}}_f = (x_f\,\hat{\imath} + y_f\,\hat{\jmath}) = [(20\,t + 2\,t^2)\,\hat{\imath} - 15\,t\,\hat{\jmath}\,] \text{ m}$$

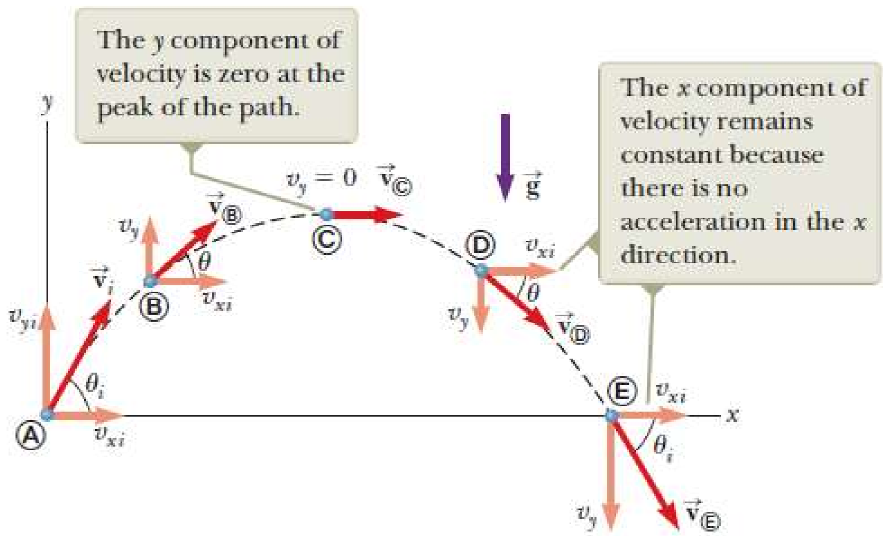
4.3 Projectile Motion

Anyone who has observed a baseball in motion has observed projectile motion. The ball moves in a curved path and returns to the ground. The path of a projectile, which we call its *trajectory,* is always a ***parabola***. The expression for the position vector of the projectile as a function of time follows directly from equation 4.9, with its acceleration being that due to gravity, $\vec{a} = \vec{g}$

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{g}}\, t^2 \qquad (4.10)$$

Where the initial x and y components of the velocity of the projectile are:

$$v_{xi} = v_i \cos\theta_i \qquad\qquad v_{yi} = v_i \sin\theta_i$$



When analyzing projectile motion, model it to be the superposition of two motions: (1) motion of a particle under constant velocity in the horizontal direction and (2) motion of a particle under constant acceleration (free fall) in the vertical direction.

Horizontal Range and Maximum Height of a Projectile

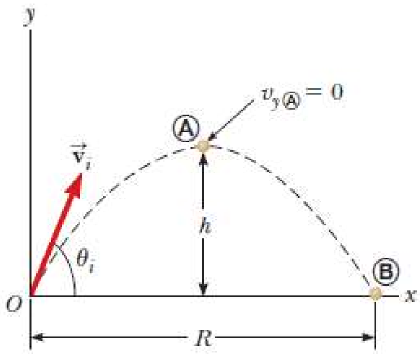
Let us assume a projectile is launched from the origin at $t_i = 0$ with a positive v_{yi} component as shown in figure above, and returns to the same horizontal level. This situation is common in sports, where baseballs, footballs, and golf balls often land at the same level from which they were launched.

Two points in this motion are especially interesting to analyze:

- The peak point A, which has Cartesian coordinates ($R/2$, h), and
- The point B, which has coordinates (R, 0).
- The distance (R) is called the **horizontal range** of the projectile, and the distance (h) is its **maximum height.**

Let us find (h) and (R) mathematically in terms of v_i, θ_i, and g :

We can determine (h) by noting that at the peak v_{yA} =0. Therefore, we can use the y component of equation (4.8) to determine the time t_A at which the projectile reaches the peak:



$$\vec{v}_{yf} = \vec{v}_{yi} + a_y\, t$$

$$0 = v_i \sin\theta\, i - g\, t_A$$

$$t_A = \frac{v_i \sin\theta i}{g}$$

Substituting this expression for t_A into the y component of equation (4.9) and replacing $y = y_A$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin\theta\, i\,)\left(\frac{v_i \sin\theta i}{g}\right) - \frac{1}{2}\, g \left(\frac{v_i^2 \sin^2\theta i}{g^2}\right)$$

$$\boxed{h = \frac{v_i^2 \sin^2\theta i}{2g}}$$ Maximum height for the projectile (4.11)

- The range R is the horizontal position of the projectile at a time that is <u>twice</u> the time at which it reaches its peak, that is, at time ($t_B = 2t_A$).

Using the x component of equation (4.9), noting that:

$v_{xi} = v_{xB} = v_i \cos\theta\, i$, and setting $x_B = R$ at $t = 2t_A$, we find that:

$R = v_{xi}\, t_B = (v_i \cos\theta\, i)\,(2t_A)$

$R = (v_i \cos\theta\, i)\left(\frac{2v_i \sin\theta i}{g}\right) = \frac{2{v_i}^2\ \sin\theta i\ \cos\theta i}{g}$

Using the identity $\sin 2\theta = 2sin\theta\ \cos\theta$, so

$\boxed{R = \frac{{v_i}^2 \sin 2\theta i}{g}}$ Horizontal range of the projectile (4.12)

The maximum value of R from equation (4.12) is:

$\boxed{R_{max} = \frac{{v_i}^2}{g}}$ because the maximum value of $(\sin 2\theta\, i = 1)$, which occurs when $2\theta i = 90°$. Therefore, <u>R is a maximum when $\theta i = 45°$.</u>

Example (4.2):

A long jumper leaves the ground at an angle of 20° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction?

(B) What is the maximum height reached?

Solution:

(A): Use equation (4.12) to find the range of the jumper:

$$R = \frac{{v_i}^2 \sin 2\theta i}{g} = \frac{(11)^2\ \sin(2X20°)}{9.8} = 7.94\ \text{m}$$

(B): The maximum height reached by using equation 4.11:

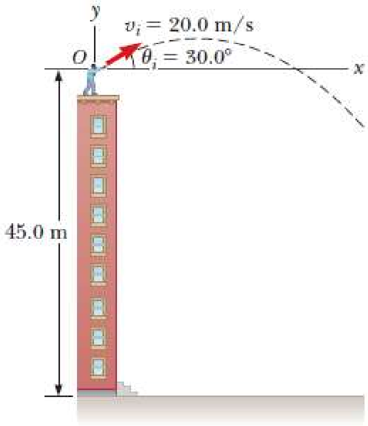
$$h = \frac{{v_i}^2\ sin^2\,\theta i}{2g} = \frac{(11)^2\ sin^2\ 20°}{2\ (9.8)} = 0.722\ \text{m}$$

Example (4.3):

A stone is thrown from the top of a building upward at an angle of (30°) to the horizontal with an initial speed of (20 m/s) as shown in the figure. The height from which the stone is thrown is (45 m) above the ground.

(A) How long does it take the stone to reach the ground?

Solution: (A) We have the information



$x_i = y_i = 0$, $y_f = -45$ m, $a_y = -$ g , and $v_i = 20$ m/s

The initial x and y components of the stone's velocity:

$v_{xi} = v_i \cos \Theta i = 20 \cos 30° = 17.3$ m/s

$v_{yi} = v_i \sin \Theta i = 20 \sin 30° = 10$ m/s

The vertical position of the stone from the vertical component:

$y_f = y_i + v_{yi}t + \frac{1}{2} a_y t^2$

$-45 = 0 + 10\, t + \frac{1}{2}(-9.8)\, t^2$

$t = 4.22$ s

(B) What is the speed of the stone just before it strikes the ground?

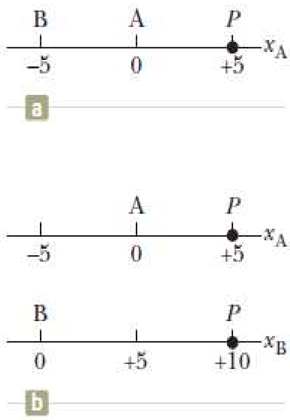
$v_{yf} = v_{yi} + a_y\, t$

$= 10 + (-9.8)(4.22) = -31.3$ m/s

4.4 Relative Velocity

We describe how observations made by different observers in different frames of reference are related to one another. A frame of reference can be described by a Cartesian coordinate system for which an observer is at rest with respect to the origin.

Consider the two observers A and B along the number line in figure a.



Observer A is located at the origin of a one-dimensional x_A axis, while observer B is at the position $x_A = -5$. We denote the position variable as x_A because observer A is at the origin of this axis. Both observers measure the position of point P, which is located at $x_A = +5$. Suppose