The two particles collide head-on and then leave the collision site with different velocities, $\vec{v}_{1f}$ and $\vec{v}_{2f}$. In an elastic collision, both the momentum and kinetic energy of the system are conserved.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{9.12}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \tag{9.13}$$

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \tag{9.14}$$

Let us separate the terms containing m_1 and m_2 in equation (9.12) to obtain:

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \tag{9.15}$$

To obtain final result, we divide equation 9.14 by equation 9.15 and obtain:

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \tag{9.16}$$

According to equation (9.16), the *relative* velocity of the two particles before the collision, $v_{1i} - v_{2i}$, equals the negative of their relative velocity after the collision, $-(v_{1f} - v_{2f})$.

Suppose the masses and initial velocities of both particles are known. Equations (9.12) and (9.16) can be solved for the final velocities in terms of the initial velocities because there are two equations and two unknowns:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i} \tag{9.17}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \tag{9.18}$$

Let us consider some special cases. If $m_1 = m_2$, equations 9.17 and 9.18 show that $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$, which means that the particles exchange velocities if they have equal masses. That is approximately what one observes in head-on billiard ball collisions: the cue ball stops and the struck ball moves away from the collision with the same velocity the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$, and equations 9.17 and 9.18 become:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \tag{9.19}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \tag{9.20}$$

Example (9.1):

A 1 800-kg car stopped at a traffic light is struck from the rear by a 900-kg car. The two cars become entangled, moving along the same path as that of the originally moving car. If the smaller car were moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution:

The phrase "become entangled" tell us that the collision is perfectly inelastic. The magnitude of the total momentum of the system before the collision is equal to that of the smaller car because the larger car is initially at rest.

Set the initial momentum of the system equal to the final momentum of the system:

$$p_i = p_f \quad \rightarrow \quad m_1 v_i = (m_1 + m_2) v_f$$

$$v_f = \frac{m_1 v_i}{m_1 + m_2} = \frac{(900\ \text{kg})(20.0\ \text{m/s})}{900\ \text{kg} + 1\ 800\ \text{kg}} = \boxed{6.67\ \text{m/s}}$$

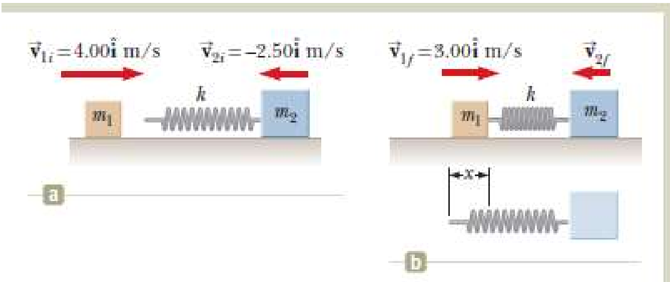
Because the final velocity is positive, the direction of the final velocity of the combination is the same as the velocity of the initially moving car. The speed of the combination is also much lower than the initial speed of the moving car.

Example (9.2):

A block of mass m_1 = 1.60 kg initially moving to the right with a speed of 4m/s on a frictionless, horizontal track collides with a light spring attached to a second block of mass m_2 = 2.10 kg initially moving to the left with a speed of 2.50 m/s as shown in the figure a. The spring constant is 600 N/m.

(A) Find the velocities of the two blocks after the collision.

Solution:



Because the spring force is conservative, kinetic energy in the system of two blocks and the spring is not transformed to internal energy during the compression of the spring. The collision is elastic.

Because momentum of the system is conserved, apply equation 9.12:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \qquad (1)$$

Because the collision is elastic, apply equation 9.16:

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \qquad (2)$$

Multiply equation (2) by m_1: $\quad m_1 v_{1i} - m_1 v_{2i} = -m_1 v_{1f} + m_1 v_{2f} \qquad (3)$

Add equations (1) and (3):

$$2m_1 v_{1i} + (m_2 - m_1)v_{2i} = (m_1 + m_2)v_{2f}$$

$$v_{2f} = \frac{2m_1 v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2}$$

$$v_{2f} = \frac{2(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg} - 1.60\text{ kg})(-2.50\text{ m/s})}{1.60\text{ kg} + 2.10\text{ kg}} = \boxed{3.12\text{ m/s}}$$

$$v_{1f} = v_{2f} - v_{1i} + v_{2i} = 3.12\text{ m/s} - 4.00\text{ m/s} + (-2.50\text{ m/s}) = \boxed{-3.38\text{ m/s}}$$

(C) Determine the velocity of block 2 during the collision, at the instant block 1 is moving to the right with a velocity of 13m/s as in figure b.

Solution:

Apply equation 9.12: $m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$

Solve for v_{2f}:

$$v_{2f} = \frac{m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}}{m_2}$$

$$v_{2f} = \frac{(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg})(-2.50\text{ m/s}) - (1.60\text{ kg})(3.00\text{ m/s})}{2.10\text{ kg}}$$

$$= \boxed{-1.74\text{ m/s}}$$

The negative value for v_{2f} means that block 2 is still moving to the left.

(D) Determine the distance the spring is compressed at that instant.

Solution:

Write a conservation of mechanical energy equation for the system:

$$K_i + U_i = K_f + U_f$$

Evaluate the energies, recognizing that two objects in the system have kinetic energy and that the potential energy is elastic:

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 + 0 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 + \tfrac{1}{2}kx^2$$

$$\tfrac{1}{2}(1.60\text{ kg})(4.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(2.50\text{ m/s})^2 + 0$$
$$= \tfrac{1}{2}(1.60\text{ kg})(3.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(1.74\text{ m/s})^2 + \tfrac{1}{2}(600\text{ N/m})x^2$$

$$x = \boxed{0.173\text{ m}}$$

Collisions in Two Dimensions

The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Where the three subscripts on the velocity components in these equations represent, respectively, the identification of the object (1, 2), initial and final values (*i, f*), and the velocity component (*x, y*).

Let us consider a specific two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2 initially at rest as in the figure. After the collision (Fig. b), particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle $\emptyset$ with respect to the horizontal. This event is called a *glancing* collision.

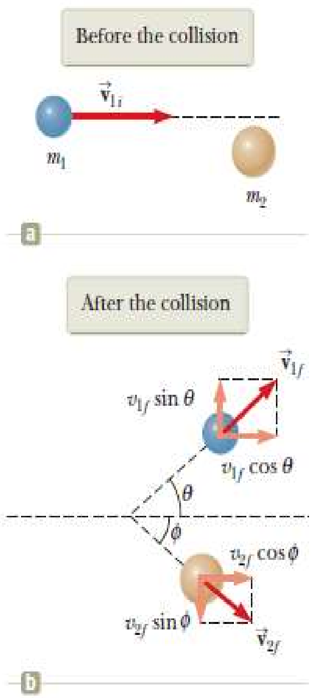
Applying the law of conservation of momentum in component form and noting that the initial *y* component of the momentum of the two-particle system is zero gives:

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi \qquad (9.21)$$
$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi \qquad (9.22)$$

Where the minus sign in equation 9.22 is included because after the collision particle 2 has a *y* component of velocity that is downward.

If the collision is elastic, we can also use equation 9.13 (conservation of kinetic energy) with $v_{2i} = 0$:

$$\tfrac{1}{2} m_1 v_{1i}{}^2 = \tfrac{1}{2} m_1 v_{1f}{}^2 + \tfrac{1}{2} m_2 v_{2f}{}^2 \qquad (9.23)$$



If the collision is inelastic, kinetic energy is *not* conserved and equation (9.23) does *not* apply.

Example (9.3):

A proton collides elastically with another proton that is initially at rest. The incoming proton has an initial speed of (3.50×10^5) m/s and makes a glancing collision with the second proton as in the figure. After the collision, one proton moves off at an angle θ =37° to the original direction of motion and the second deflect at an angle Ø to the same axis. Find the final speeds of the two protons and the angle Ø.

Solution:

Both momentum and kinetic energy of the system are conserved in this glancing elastic collision.

Use equation (9.21) through equation (9.23) gives:

$$v_{1f}\cos\theta + v_{2f}\cos\phi = v_{1i} \qquad (1)$$

$$v_{1f}\sin\theta - v_{2f}\sin\phi = 0 \qquad (2)$$

$$v_{1f}{}^2 + v_{2f}{}^2 = v_{1i}{}^2 \qquad (3)$$

Rearrange equations (1) and (2):

$$v_{2f}\cos\phi = v_{1i} - v_{1f}\cos\theta$$

$$v_{2f}\sin\phi = v_{1f}\sin\theta$$

Square these two equations and add them:

$$v_{2f}{}^2\cos^2\phi + v_{2f}{}^2\sin^2\phi = v_{1i}{}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}{}^2\cos^2\theta + v_{1f}{}^2\sin^2\theta$$

Since $sin^2\theta + cos^2\theta$ =1

$$v_{2f}{}^2 = v_{1i}{}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}{}^2 \qquad (4)$$

Substitute equation (4) into equation (3):

$$v_{1f}{}^2 + (v_{1i}{}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}{}^2) = v_{1i}{}^2$$

$$v_{1f}{}^2 - v_{1i}v_{1f}\cos\theta = 0 \qquad (5)$$

One possible solution of equation (5) is $v_{1f} = 0$, which corresponds to a head-on, one-dimensional collision in which the first proton stops and the second continues with the same speed in the same direction. That is not the solution we want.

Divide both sides of equation (5) by v_{1f} and solve for the remaining factor of v_{1f} :

$$v_{1f} = v_{1i} \cos\theta = (3.50 \times 10^5 \text{ m/s}) \cos 37.0° = 2.80 \times 10^5 \text{ m/s}$$

Use equation (3) to find v_{2f} :

$$v_{2f} = \sqrt{v_{1i}^2 - v_{1f}^2} = \sqrt{(3.50 \times 10^5 \text{ m/s})^2 - (2.80 \times 10^5 \text{ m/s})^2}$$
$$= 2.11 \times 10^5 \text{ m/s}$$

Use equation (2) to find $\emptyset$:

$$\phi = \sin^{-1}\left(\frac{v_{1f} \sin\theta}{v_{2f}}\right) = \sin^{-1}\left[\frac{(2.80 \times 10^5 \text{ m/s}) \sin 37.0°}{(2.11 \times 10^5 \text{ m/s})}\right]$$
$$= 53.0°$$

It is interesting that ($\theta + \emptyset = 90°$). This result is *not* accidental. Whenever two objects of equal mass collide elastically in a glancing collision and one of them is initially at rest, their final velocities are perpendicular to each other.

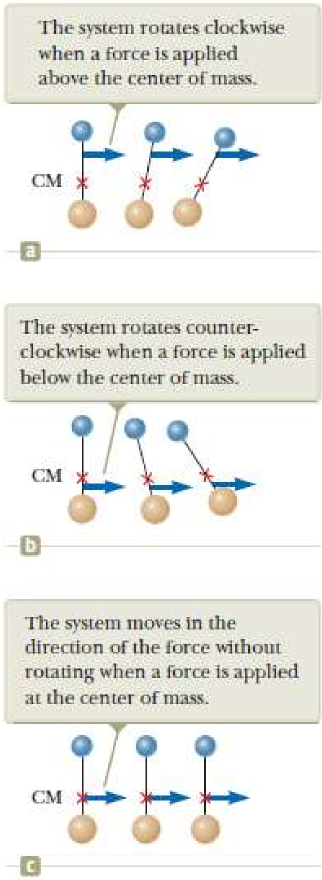
9.3 The Center of Mass

We describe the overall motion of a system in terms of a special point called the **center of mass** of the system. The system can be a group of particles, such as a collection of atoms in a container.

Consider a system consisting of a pair of particles that have different masses and are connected by a light, rigid rod as shown in the figure.

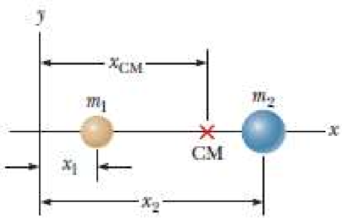
The position of the center of mass of a system can be described as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass.

If a single force is applied at a point on the rod above the center of mass, the system rotates clockwise as shown in the figure (a).
If the force is applied at a point on the rod below the center of mass, the system rotates counterclockwise as shown in the figure (b).
If the force is applied at the center of mass, the system moves in the direction of the force without rotating as shown in the figure (c).



The center of mass of the pair of particles described in the figure below is located on the *x* axis and lies somewhere between the particles. Its *x* coordinate is given by:

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad (9.24)$$



- For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = 2/3\ d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.
- We can extend this concept to a system of many particles with masses m_i in three dimensions.

The x - coordinate of the center of mass of (n) particles is defined to be:

$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{1}{M}\sum_i m_i x_i \qquad (9.25)$$

Where x_i is the x coordinate of the i^{th} particle and the total mass is

$$M = \sum_i m_i$$

The center of mass can be located in three dimensions by its position vector $\vec{r}_{CM}$.

$$\vec{\mathbf{r}}_{CM} \equiv \frac{1}{M}\sum_i m_i \vec{\mathbf{r}}_i \qquad (9.26)$$

Where $\vec{r}_i$ is the position vector of the i^{th} particle, defined by:

$$\vec{\mathbf{r}}_i \equiv x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

- We replace the sum by an integral and (Δm_i) by the differential (dm):

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M}\sum_i x_i \Delta m_i = \frac{1}{M}\int x\, dm$$

$$x_{CM} = \frac{1}{M}\int x\, dm \qquad (9.27)$$

- We can express the vector position of the center of mass:

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M}\int \vec{\mathbf{r}}\, dm \qquad (9.28)$$

- The center of mass of a uniform rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

Example (9.4):

A system consists of three particles located as shown in the figure. Find the center of mass of the system. The masses of the particles are $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg.