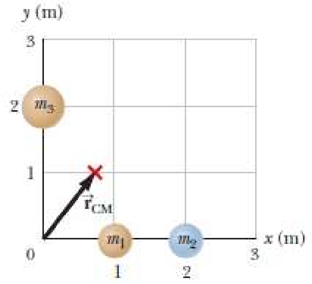


Solution:

$$x_{CM} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(1.0\text{ m}) + (1.0\text{ kg})(2.0\text{ m}) + (2.0\text{ kg})(0)}{1.0\text{ kg} + 1.0\text{ kg} + 2.0\text{ kg}} = \frac{3.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 0.75\text{ m}$$

$$y_{CM} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(0) + (1.0\text{ kg})(0) + (2.0\text{ kg})(2.0\text{ m})}{4.0\text{ kg}} = \frac{4.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 1.0\text{ m}$$

$$\vec{r}_{CM} \equiv x_{CM}\hat{i} + y_{CM}\hat{j} = \boxed{(0.75\hat{i} + 1.0\hat{j})\text{ m}}$$

Example (9.5):

(A) Show that the center of mass of a rod of mass (M) and length (L) lies midway between its ends, assuming the rod has a uniform mass per unit length.

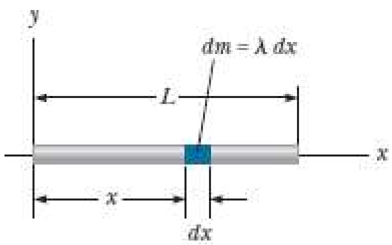
Solution:

The mass per unit length (this quantity is called the *linear mass density*) can be written as $\lambda = M/L$ for the uniform rod. If the rod is divided into elements of length dx, the mass of each element is $dm = \lambda\, dx$.

Use equation (9.27) to find an expression for x_{CM}:

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \boxed{\tfrac{1}{2}L}$$



(B) Suppose a rod is *non-uniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Solution: In this case, we replace (dm) in equation (9.27) by ($\lambda\, dx$), where $\lambda = \alpha\, x$.

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\tfrac{2}{3}L}$$

Notice that the center of mass in part (B) is farther to the right than that in part (A).

Chapter 10

(Rotational motion)

10.1 Angular Position, Velocity, and Acceleration

Figure (10.1) illustrates a rotating compact disc, or CD. The disc rotates about a fixed axis perpendicular to the plane of the figure and passing through the center of the disc at *O*. A small element of the disc modeled as a particle at *P* is at a fixed distance (*r*) from the origin and rotates about it in a circle of radius *r*. (In fact, every element of the disc undergoes circular motion about *O*). It is convenient to represent the position of (*P*) with its polar coordinates (*r*, θ), where *r* is the distance from the origin to *P* and θ is measured *counterclockwise* from some reference line fixed in space as shown in figure (10.1a).

In this representation, the angle θ changes in time while *r* remains constant. As the particle moves along the circle from the reference line, which is at angle θ = 0, it moves through an arc of length *s* as in figure (10.1b).

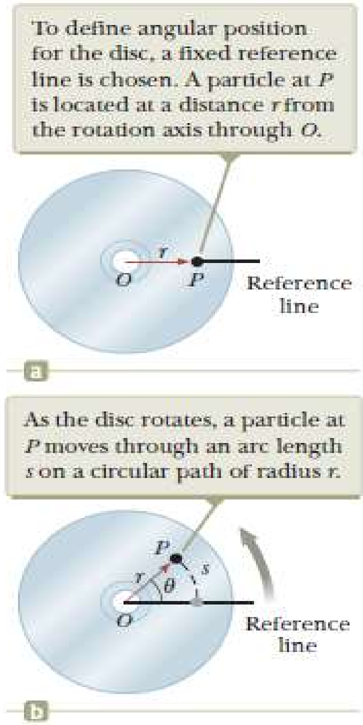


Figure 10.1 A compact disc rotating about a fixed axis through *O* perpendicular to the plane of the figure.

The arc length *s* is related to the angle θ through the relationship:

$$s = r\,\theta \qquad (10.1)$$

$$\theta = \frac{s}{r}$$

Because θ is the ratio of an arc length and the radius of the circle, it is a pure number.

We give θ the artificial unit **radian** (rad).

- Because the circumference of a circle is $2\pi r$,

it follows from equation (10.1) that:

360° corresponds to an angle of $(2\pi r/r)$ rad= 2π rad.

Hence, 1 rad = 360°/2π = 57.3°.

- To convert an angle in degrees to an angle in radians, we use :

$$\pi \text{ rad} = 180°$$

$$\text{so,} \qquad \theta\,(\text{ rad }) = \pi / 180°\,(\text{ degree })$$

For example 60° = (π / 3) rad and 45° = (π / 4) rad.

- We choose a reference line on the object, such as a line connecting *O* and a chosen particle on the object.

  The **angular position** of the rigid object is (the angle θ between this reference line on the object and the fixed reference line in space), which is often chosen as the *x* - axis.

- As the particle travels from position (A) to position (B) in a time interval Δt as in figure (10.2), the reference line fixed to the object sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity Δθ is defined as the **angular displacement** of the rigid object:

$$\Delta\theta = \theta_f - \theta_i$$

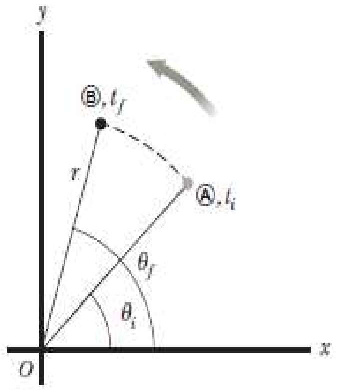


Figure 10.2 A particle on a rotating rigid object moves from Ⓐ to Ⓑ along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radial line of length *r* moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$.

- The **average angular speed** (ω_{avg}) (Greek letter omega) as (the ratio of the angular displacement of a rigid object to the time interval Δt during which the displacement occurs):

$$\omega_{avg} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \textbf{Average angular speed} \qquad (10.2)$$

- The **instantaneous angular speed** ω is defined as (the limit of the average angular speed as Δt approaches zero):

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$ **Instantaneous angular speed** (10.3)

Angular speed has units of radians per second (rad/s), which can be written as (s^{-1}) because radians are not dimensional.

- We take (ω) to be positive when θ is increasing (counterclockwise motion in figure 10.2) and negative when θ is decreasing (clockwise motion in figure 10.2).
- If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** α_{avg} (Greek letter alpha) of a rotating rigid object is defined as (the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs):

$$\alpha_{avg} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t}$$ **Average angular acceleration** (10.4)

The **instantaneous angular acceleration** is defined as the limit of the average angular acceleration as Δt approaches zero:

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$ **Instantaneous angular acceleration** (10.5)

- Angular acceleration has units of radians per second squared (rad/s^2), or simply (s^{-2}).
- Notice that (α) is positive when a rigid object rotating counterclockwise is speeding up or when a rigid object rotating clockwise is slowing down during some time interval.

- It is convenient to use the *right-hand rule* demonstrated in figure (10.3) to determine the direction of $\vec{\omega}$:

  (When the four fingers of the right hand are wrapped in the direction of rotation, the extended right thumb points in the direction of $\vec{\omega}$).
- The direction of $\vec{\alpha}$ follows from its definition ($\vec{\alpha} \equiv d\vec{\omega}/dt$). It is in the same direction as $\vec{\omega}$ if the angular speed is increasing in time, and it is antiparallel to $\vec{\omega}$ if the angular speed is decreasing in time.

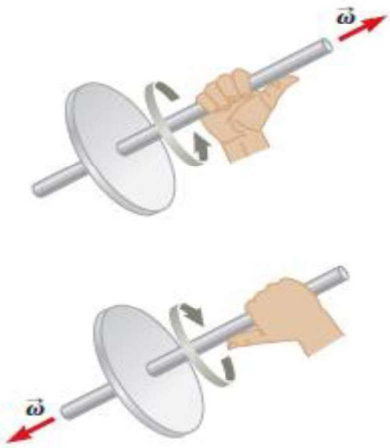


Figure 10.3 The right-hand rule for determining the direction of the angular velocity vector.

10.2 Rigid Object under Constant Angular Acceleration

Writing equation (10.5) in the form: ($d\omega = \alpha dt$) and integrating from $t_i = 0$ to $t_f = t$ gives:

$$\omega_f = \omega_i + \alpha t \qquad \text{for constant} \qquad (10.6)$$

Where ω_i is the angular speed of the rigid object at time $t = 0$. Equation (10.6) allows us to find the angular speed ω_f of the object at any later time t. Substituting equation (10.6) into equation (10.3) and integrating once more, we obtain:

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \quad \text{(for constant } \alpha) \qquad (10.7)$$

Where θ_i is the angular position of the rigid object at time $t = 0$.

Equation (10.7) allows us to find the angular position θ_f of the object at any later time t.

Eliminating (t) from equations (10.6) and (10.7) gives:

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \quad \text{(for constant } \alpha) \qquad (10.8)$$

This equation allows us to find the angular speed ω_f of the rigid object for any value of its angular position θ_f.

If we eliminate (α) between equations (10.6) and (10.7), we obtain:

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)t \quad \text{(for constant } \alpha) \qquad (10.9)$$

Notice that these kinematic expressions for the rigid object under constant angular acceleration are of the same mathematical form as those for a particle under constant acceleration (Chapter 2).

They can be generated from the equations for translational motion by making the substitutions $x \rightarrow \theta$, $v \rightarrow \omega$, and $a \rightarrow \alpha$.

Table (10.1) compares the kinematic equations for rotational and translational motion.

TABLE 10.1 *Kinematic Equations for Rotational and Translational Motion*

Rigid Body Under Constant Angular Acceleration	Particle Under Constant Acceleration
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Example (10.1):

A wheel rotates with a constant angular acceleration of ($3.50\ rad/s^2$).

(A) If the angular speed of the wheel is (2 rad/s) at $t_i = 0$, through what angular displacement does the wheel rotate in 2 s?

Solution: Arrange equation (10.7) so that it expresses the angular displacement of the object: $\Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$

Substitute the known values to find the angular displacement at t =2 s:

$$\Delta\,\theta = (2\ rad/s)(2\ s) + \ 1/2\ (3.5\ rad/s^2\ )\ (\ 2\ s\ )^2 = 11\ rad$$

$$= (11\ rad\ )(\ 180°/\pi\ rad) = 630°$$

(B): Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630°\left(\frac{1\ \text{rev}}{360°}\right) = 1.75\ \text{rev}$$

(C): What is the angular speed of the wheel at t =2 s?

$$\omega_f = \omega_i + \alpha t = 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00\ \text{s})$$
$$= 9.00\ \text{rad/s}$$

10.3 Angular and Translational Quantities

Point P in figure (10.4) moves in a circle, the translational velocity vector $\vec{\mathbf{v}}$ is always tangent to the circular path and hence is called *tangential velocity.* The magnitude of the tangential velocity of the point P is by definition the tangential speed ($v = ds/dt$), where (s) is the distance traveled by this point measured along the circular path.

Recalling that $s = r\,\theta$ and noting that r is constant,

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because $d\,\theta/dt = \omega$

Then $v = r\,\omega$ (10.10)

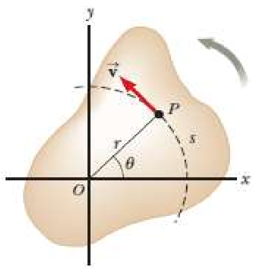


Figure (10.4)

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a_t = r\alpha$$ Relation between tangential acceleration and angular acceleration (10.11)

We can express the centripetal acceleration at that point in terms of angular speed as:

$$a_c = \frac{v^2}{r} = r\omega^2 \qquad (10.12)$$

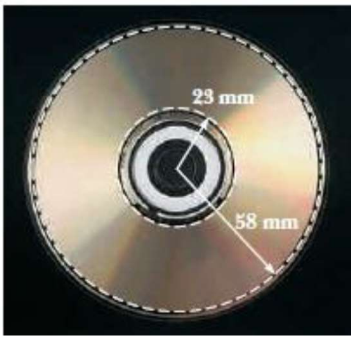
The total acceleration vector at the point P is $\vec{\mathbf{a}} = \vec{\mathbf{a}}_t + \vec{\mathbf{a}}_r$, where the magnitude of $\vec{\mathbf{a}}_r$ is the centripetal acceleration a_c. Because $\vec{\mathbf{a}}$ is a vector having a radial and a tangential component, the magnitude of $\vec{\mathbf{a}}$ at the point P on the rotating rigid object is:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$ Total Acceleration) (10.13)

Example (10.2):

(A) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track (r = 23 mm) and the outermost final track (r =58 mm)? The constant speed of the CD player is 1.3 m/s.

Solution:



The angular speed that gives the required tangential speed at the position of the inner track:

$$\omega_i = \frac{v}{r_i} = \frac{1.3\ \text{m/s}}{2.3 \times 10^{-2}\ \text{m}} = 57\ \text{rad/s}$$

$$= (57\ \text{rad/s})\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = \boxed{5.4 \times 10^2\ \text{rev/min}}$$