The same for the outer track:

$$\omega_f = \frac{v}{r_f} = \frac{1.3 \text{ m/s}}{5.8 \times 10^{-2} \text{ m}} = 22 \text{ rad/s} = \boxed{2.1 \times 10^2 \text{ rev/min}}$$

(B) The maximum playing time of a standard music disc is 74 min and 33 s. How many revolutions does the disc make during that time?

Solution:

If t = 0 is the instant the disc begins rotating, with angular speed of 57 rad/s, the final value of the time t is:

[(74 min)(60 s/min) + 33 s] =4473 s.

The angular displacement $\Delta\theta$ during this time interval is:

$$\Delta\theta = \theta_f - \theta_i = \tfrac{1}{2}(\omega_i + \omega_f)t$$
$$= \tfrac{1}{2}(57 \text{ rad/s} + 22 \text{ rad/s})(4\,473 \text{ s}) = 1.8 \times 10^5 \text{ rad}$$

Convert this angular displacement to revolutions:

$$\Delta\theta = (1.8 \times 10^5 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = \boxed{2.8 \times 10^4 \text{ rev}}$$

(C) What is the angular acceleration of the compact disc over the 4473 s time interval?

Solution:

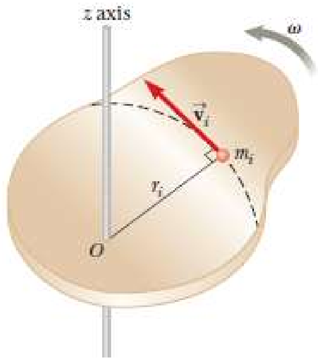
$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22 \text{ rad/s} - 57 \text{ rad/s}}{4\,473 \text{ s}} = \boxed{-7.6 \times 10^{-3} \text{ rad/s}^2}$$

10.4 Rotational Kinetic Energy

Let us consider an object as a system of particles and assume it rotates about a fixed z- axis with an angular speed ω. Figure (10.5) shows the rotating object and identifies one particle on the object located at a distance r_i from the rotation axis. If the mass of the i^{th} particle is m_i and its tangential speed is (v_i), its kinetic energy is:

$$K_i = \frac{1}{2} m_i v_i^2$$

The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:



$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

We can write this expression in the form:

$$K_R = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2 \qquad (10.14)$$

Figure (10.5)

Where we have factored (ω^2) from the sum because it is common to every particle. We simplify this expression by defining the quantity in parentheses as the **moment of inertia *I*** of the rigid object:

$$\boxed{I \equiv \sum_i m_i r_i^2} \qquad \textbf{Moment of inertia} \qquad (10.15)$$

It has dimensions ($Kg.m^2$).

Equation (10.15) becomes:

$$\boxed{K_R = \frac{1}{2} I \omega^2} \qquad \textbf{Rotational kinetic energy} \qquad (10.16)$$

The analogy between kinetic energy ($\frac{1}{2} m v^2$) associated with translational motion and rotational kinetic energy ($\frac{1}{2} I \omega^2$). The quantities I and ω in rotational motion are analogous to m and v in translational motion, respectively.

- Moment of inertia is (a measure of the resistance of an object to changes in its rotational motion), just as mass is (a measure of the of the tendency of an object to resist changes in its translational Motion).

10.5 Calculation of Moments of Inertia

We use the definition ($I = \Sigma_i\ r_i^2 \Delta m_i$) and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

Moment of inertia of a rigid object:

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2\ \Delta m_i = \int r^2\ dm \qquad (10.17)$$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using:

$\rho = \frac{m}{V}$ where ρ is the density of the object and V is its volume.

The mass of small element is: $dm = \rho\ dV$

Substituting this result into equation (10.17) gives:

$$\mathrm{I} = \int \rho\ r^2\ dV$$

If the object is homogeneous, ρ is constant.

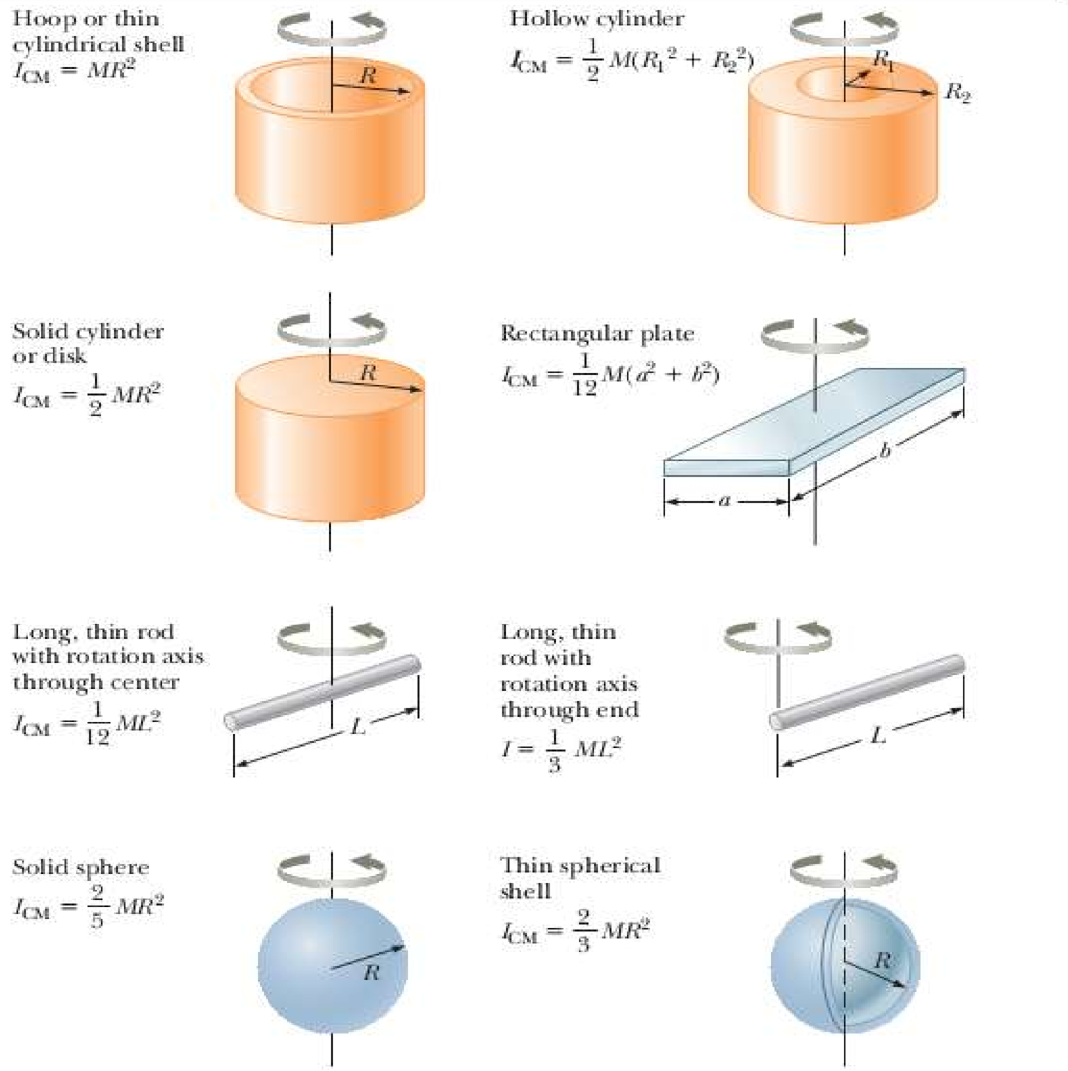
The density given by ($\rho = m/V$) sometimes is referred to as (*volumetric mass density*) because it represents (*mass per unit volume*).

For instance, when dealing with a sheet of uniform thickness t, we can define a (*surface mass density*) ($\sigma = \rho t$), which represents (*mass per unit area*).

Finally, when mass is distributed along a rod of uniform cross-sectional area A, we sometimes use (*linear mass density*) ($\lambda = M\,/\,L = \rho\ A$), which is the (*mass per unit length*).

Table (10.2) gives the moments of inertia for a number of objects about specific axes.

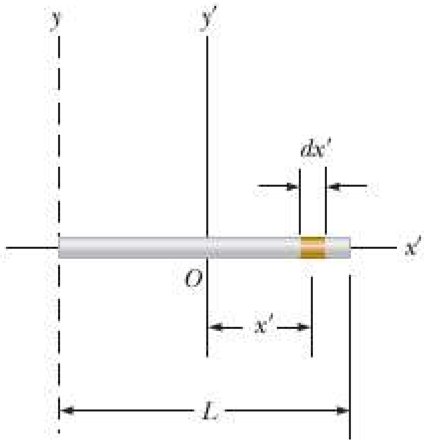
TABLE 10.2 *Moments of Inertia of Homogeneous Rigid Objects with Different Geometries*



Example (10.3):

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass?

Solution:



The shaded length element dx' in the figure has a mass dm equal to the mass per unit length λ multiplied by dx'.

Express dm in terms of dx':

$$dm = \lambda\, dx' = \frac{M}{L}\, dx'$$

Substitute this expression into equation (10.17), with $r^2 = (x')^2$

$$I_y = \int r^2\, dm = \int_{-L/2}^{L/2} (x')^2 \frac{M}{L}\, dx' = \frac{M}{L}\int_{-L/2}^{L/2} (x')^2\, dx'$$

$$= \frac{M}{L}\left[\frac{(x')^3}{3}\right]_{-L/2}^{L/2} = \boxed{\tfrac{1}{12}ML^2}$$

Moment of inertia for thin rod

Check this result in Table (10.2).

Example (10.4):

A uniform solid cylinder has a radius R, mass M, and length L.

Calculate its moment of inertia about its central axis?

Solution:

It is convenient to divide the cylinder into many cylindrical shells, each having radius r, thickness dr, and length L as shown in the figure.

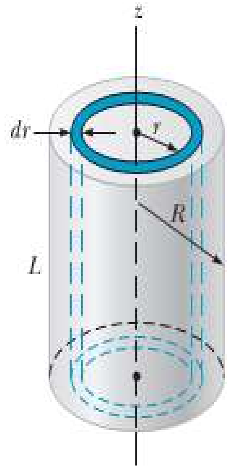
The density of the cylinder is ρ.

The volume dV of each shell is its cross-sectional area multiplied by its length:

$$dV = L\,dA = L(2\pi r)\,dr$$

Express dm in terms of dr:

$$dm = \rho\,dV = \rho\,L(2\pi r)\,dr$$



Substitute this expression into equation (10.17):

$$I_z = \int r^2\,dm = \int r^2[\rho L(2\pi r)\,dr] = 2\pi\rho L\int_0^R r^3\,dr = \tfrac{1}{2}\pi\rho L R^4$$

Use the total volume ($\pi R^2 L$) of the cylinder to express its density:

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$$

Substitute this value into the expression for I_z :

$$I_z = \tfrac{1}{2}\pi\left(\frac{M}{\pi R^2 L}\right)LR^4 = \tfrac{1}{2}MR^2$$

Moment of inertia for a cylinder

Check this result in Table (10.2).

Notice that the result for the moment of inertia of a cylinder does not depend on L, the length of the cylinder. Therefore, the moment of inertia of the cylinder would not be affected by changing its length.

Parallel-axis theorem:

$$I = I_{CM} + MD^2 \qquad \textbf{Parallel-axis theorem} \qquad (10.18)$$

I_{CM} : The moment of inertia about an axis that is parallel to the z - axis and passes through the center of mass.

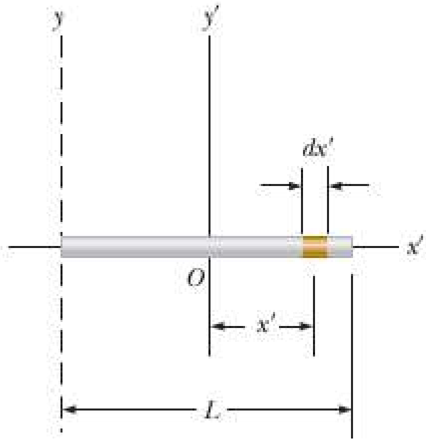
D is the distance between the center of mass axis and an axis parallel to that axis.

Example (10.5):

Find the moment of inertia of uniform rigid rod of mass M and length L about an axis perpendicular to the rod through one end (the y axis in the figure)?

Solution:

The distance between the center of mass axis and the y - axis is $D = L/2$.



Use the parallel-axis theorem:

$$I = I_{CM} + MD^2 = \tfrac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \tfrac{1}{3}ML^2$$

Check this result in Table (10.2).

10.6 Torque

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis.

- (The tendency of a force to rotate an object about some axis is measured by a quantity called **torque** $\vec{\tau}$) (Greek letter tau). Torque is a vector.

  Consider the wrench in the figure (10.6) that we wish to rotate around an axis that is perpendicular to the page and passes through the center of the bolt.

  The applied force $\vec{F}$ acts at an angle ϕ to the horizontal.

  We define the magnitude of the torque associated with the force $\vec{F}$ around the axis passing through O by the expression:

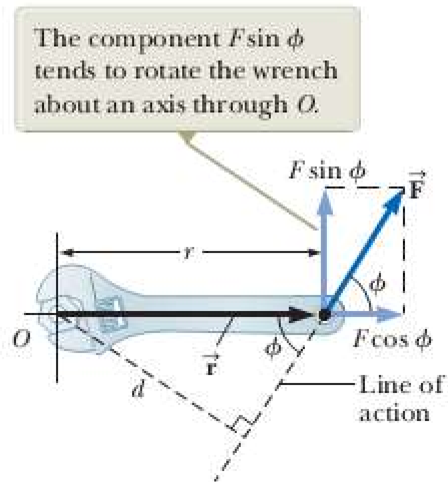


Figure (10.6)

$$\tau = rF\sin\phi = Fd \qquad (10.19)$$

where r is the distance between the rotation axis and the point of application of $\vec{F}$ and d is the perpendicular distance from the rotation axis to the line of action of $\vec{F}$.

(The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force). The dashed line extending from the tail of $\vec{F}$ in figure (10.6) is part of the line of action of $\vec{F}$. From the right triangle in figure (10.6) that has the wrench as its hypotenuse, we see that

$d = r\, sin\, \phi$ The quantity d is called the **moment arm** of $\vec{F}$.

- The only component of $\vec{F}$ that tends to cause rotation of the wrench around an axis through O is ($F\ sin\ \phi$) (the component perpendicular to a line drawn from the rotation axis to the point of application of the force).
- The horizontal component ($Fcos\ \phi$), because its line of action passes through O, has no tendency to produce rotation about an axis passing through O.
- From the definition of torque, the rotating tendency increases as F increases and as d increases.
- Torque has units of force times length (Newton.meter).
- If the torque is positive, the object begins to rotate in the counterclockwise direction and if it is negative, the rotation is clockwise.

10.7 Rigid Object under a Net Torque

The rotational analog of Newton's second law: (The angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis).

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential net force $\Sigma\,\vec{F}_t$ and a radial net force $\Sigma\,\vec{F}_r$ as shown in figure (10.7).

The radial net force causes the particle to move in the circular path with a centripetal acceleration. The tangential force provides a tangential acceleration $\vec{a}_t$, and

$$\Sigma\,\vec{F}_t = m\vec{a}_t$$

The magnitude of the net torque due to $\Sigma\,\vec{F}_t$ on the particle about an axis perpendicular to the page through the center of the circle is:

$$\Sigma\,\tau = \Sigma\,F_t\,r = (ma_t\,)\,r$$

Because the tangential acceleration is related to the angular acceleration through the relationship : $a_t = r\alpha$

The net torque can be expressed as:

$$\Sigma\,\tau = (mr\alpha\,)\,r = (\,mr^2\,)\,\alpha$$

Since $(\,mr^2\,)$ is the moment of inertia of the particle about the z-axis passing through the origin, so

$$\Sigma\,\tau = I\,\alpha \qquad (10.20)$$

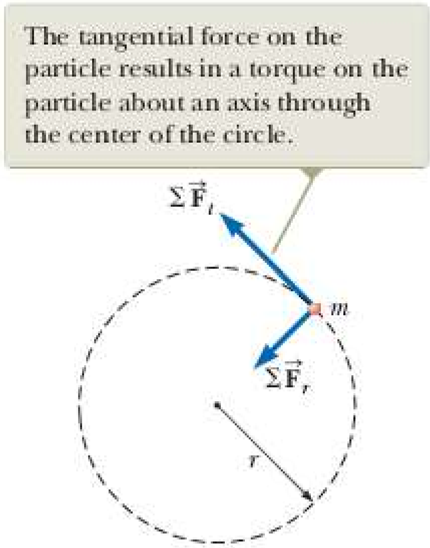
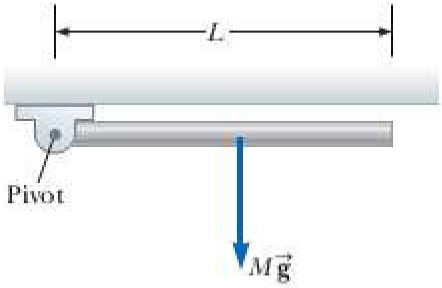


Figure (10.7)

That is, the net torque acting on the particle is proportional to its angular acceleration, and the proportionality constant is the moment of inertia. Notice that ($\Sigma\,\tau = I\,\alpha$) has the same mathematical form as Newton's second law of motion, ($\Sigma\,F = ma$).

Example (10.6):

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane as in the figure. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial translational acceleration of its right end?



Solution:

When the rod released, it rotates clockwise around the pivot at the left end. The only force contributing to the torque about an axis through the pivot is the gravitational force ($M\vec{g}$) exerted on the rod.

$$\tau = Mg\,(L/2)$$

To obtain the angular acceleration of the rod:

$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

To find the initial translational acceleration of the right end of the rod:

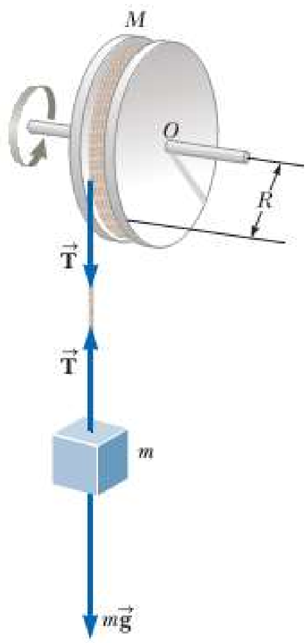
$$a_t = r\,\alpha \qquad \text{with } r = L$$

$$a_t = \tfrac{3}{2}g$$

Example (10.7):

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as in the figure. A light cord wrapped around the wheel supports an object of mass m. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates with an angular acceleration. Calculate the angular acceleration of the wheel, the translational acceleration of the object, and the tension in the cord?

Solution:



The magnitude of the torque acting on the wheel about its axis of rotation is:

$$\tau = TR$$

Where T is the force exerted by the cord on the rim of the wheel.

Use equation (10.20): $\Sigma\,\tau = I\,\alpha$

Then $\alpha = \frac{\Sigma\,\tau}{I} = \frac{TR}{I}$ (1)

Apply Newton's second law to the motion of the object, taking the downward direction to be positive:

$$\Sigma\,F_y = mg - T = ma$$

The acceleration is: $a = \frac{mg - T}{m}$ (2)

Therefore, the angular acceleration (α) of the wheel and the translational acceleration of the object are related by: $a = R\,\alpha$

Use this fact together with equations (1) and (2):

$$a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m} \qquad (3)$$

The tension T is: $T = \frac{mg}{1 + (mR^2/I)}$ (4)

Substitute equation (4) into equation (2) and solve for a:

$$a = \frac{g}{1 + (I/mR^2)} \qquad (5)$$

Use ($a = R\,\alpha$) and equation (5) to solve for a:

$$\alpha = \frac{a}{R} = \frac{g}{R + (I/mR)}$$