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Infinite Series and Power Series

THE GEOMETRIC SERIES

As a simple example of many of the ideas involved in series, we are going to consider the geometric series. You may recall that in a geometric progression we multiply each term by some fixed number to get the next term. For example, the *sequences*

$$2,\ 4,\ 8,\ 16,\ 32, \ldots, \tag{1.1a}$$

$$1,\ \frac{2}{3},\ \frac{4}{9},\ \frac{8}{27},\ \frac{16}{81}, \ldots, \tag{1.1b}$$

$$a,\ ar,\ ar^2,\ ar^3, \ldots, \tag{1.1c}$$

are geometric progressions. It is easy to think of examples of such progressions. Suppose the number of bacteria in a culture doubles every hour. Then the terms of (1.1a) represent the number by which the bacteria population has been multiplied after 1 hr, 2 hr, and so on. Or suppose a bouncing ball rises each time to $\frac{2}{3}$ of the height of the previous bounce. Then (1.1b) would represent the heights of the successive bounces in yards if the ball is originally dropped from a height of 1 yd.

This expression is an example of an *infinite series*, and we are asked to find its sum. Not all infinite series have sums; you can see that the series formed by adding the terms in (1.1a) does not have a finite sum. However, even when an infinite series does have a finite sum, we cannot find it by adding the terms because no matter how many we add there are always more. Thus we must find another method. (It is actually deeper than this; what we really have to do is to *define* what we mean by the sum of the series.)

Let us first find the sum of n terms in (1.3). The formula (Problem 2) for the sum of n terms of the geometric progression (1.1c) is

$$S_n = \frac{a(1-r^n)}{1-r}. \tag{1.4}$$

Using (1.4) in (1.3), we find

$$S_n = \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}[1-(\frac{2}{3})^n]}{1-\frac{2}{3}} = 2\left[1-\left(\frac{2}{3}\right)^n\right]. \tag{1.5}$$

Series such as (1.3) whose terms form a geometric progression are called *geometric series.* We can write a geometric series in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots . \tag{1.6}$$

The sum of the geometric series (if it has one) is by definition

$$S = \lim_{n\to\infty} S_n, \tag{1.7}$$

where S_n is the sum of n terms of the series. By following the method of the example above, you can show (Problem 2) that a geometric series has a sum if and only if $|r| < 1$, and in this case the sum is

$$S = \frac{a}{1-r}. \tag{1.8}$$

Example:

Here is an interesting use of (1.8). We can write $0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3/10}{1-1/10} = \frac{1}{3}$ by (1.8). Now of course you knew that, but how about $0.785714285714\cdots$? We can write this as $0.5 + 0.285714285714\cdots = \frac{1}{2} + \frac{0.285714}{1-10^{-6}} = \frac{1}{2} + \frac{285714}{999999} = \frac{1}{2} + \frac{2}{7} = \frac{11}{14}$.

Problems

1. In the bouncing ball example above, find the height of the tenth rebound, and the distance traveled by the ball after it touches the ground the tenth time. Compare this distance with the total distance traveled.

2. Derive the formula (1.4) for the sum S_n of the geometric progression $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. *Hint:* Multiply S_n by r and subtract the result from S_n; then solve for S_n. Show that the geometric series (1.6) converges if and only if $|r| < 1$; also show that if $|r| < 1$, the sum is given by equation (1.8).

Use equation (1.8) to find the fractions that are equivalent to the following repeating decimals:

3. $0.55555\cdots$

4. $0.818181\cdots$

5. $0.583333\cdots$

6. $0.61111\cdots$

7. $0.185185\cdots$

8. $0.694444\cdots$

9. $0.857142857142\cdots$

10. $0.576923076923076923\cdots$

11. $0.678571428571428571\cdots$

DEFINITIONS AND NOTATION

There are many other infinite series besides geometric series. Here are some examples:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots , \tag{2.1a}$$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots , \tag{2.1b}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots . \tag{2.1c}$$

In general, an infinite series means an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots , \tag{2.2}$$

where the a_n's (one for each positive integer n) are numbers or functions given by some formula or rule. The three dots in each case mean that the series never ends.

We can also write series in a shorter abbreviated form using a summation sign $\sum$ followed by the formula for the nth term. For example, (2.3a) would be written

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = \sum_{n=1}^{\infty} n^2 \tag{2.4}$$

(read "the sum of n^2 from $n = 1$ to ∞"). The series (2.3b) would be written

$$x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!}$$

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).