Theorems about Power Series

THEOREMS ABOUT POWER SERIES

We have seen that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges in some interval with center at the origin. For each value of x (in the interval of convergence) the series has a finite sum whose value depends, of course, on the value of x. Thus we can write the sum of the series as $S(x) = \sum_{n=0}^{\infty} a_n x^n$. We see then that a power series (within its interval of convergence) defines a function of x, namely $S(x)$. In describing the relation of the series and the function $S(x)$, we may say that the series converges to the function $S(x)$, or that the function $S(x)$ is represented by the series, or that the series is the power series of the function. Here we have thought of obtaining the function from a given series. We shall also (Section 12) be interested in finding a power series that converges to a given function. When we are working with power series and the functions they represent, it is useful to know the following theorems (which we state without proof; see advanced calculus texts). Power series are very useful and convenient because within their interval of convergence they can be handled much like polynomials.

1. A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series within the same interval of convergence as the original series (that is, not necessarily at the endpoints of the interval).

2. Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence. You may divide two series if the denominator series is not zero at $x = 0$, or if it is and the zero is canceled by the numerator [as, for example, in $(\sin x)/x$; see (13.1)]. The resulting series will have *some* interval of convergence (which can be found by the ratio test or more simply by complex variable theory—see Chapter 2, Section 7).

3. One series may be substituted in another provided that the values of the substituted series are in the interval of convergence of the other series.

4. The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

EXPANDING FUNCTIONS IN POWER SERIES

Very often in applied work, it is useful to find power series that represent given functions. We illustrate one method of obtaining such series by finding the series for $\sin x$. In this method we *assume* that there *is* such a series (see Section 14 for discussion of this point) and set out to find what the coefficients in the series must be. Thus we write

$$\sin x = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{12.1}$$

and try to find numerical values of the coefficients a_n to make (12.1) an identity (within the interval of convergence of the series). Since the interval of convergence of a power series contains the origin, (12.1) must hold when $x = 0$. If we substitute $x = 0$ into (12.1), we get $0 = a_0$ since $\sin 0 = 0$ and all the terms except a_0 on the right-hand side of the equation contain the factor x. Then to make (12.1) valid at $x = 0$, we must have $a_0 = 0$. Next we differentiate (12.1) term by term to get

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots . \tag{12.2}$$

(This is justified by Theorem 1 of Section 11.) Again putting $x = 0$, we get $1 = a_1$. We differentiate again, and put $x = 0$ to get

$$\begin{aligned} -\sin x &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots , \\ 0 &= 2a_2. \end{aligned} \tag{12.3}$$

Continuing the process of taking successive derivatives of (12.1) and putting $x = 0$, we get

$$\begin{aligned} -\cos x &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots , \\ -1 &= 3!\, a_3, \qquad a_3 = -\frac{1}{3!}; \\ \sin x &= 4 \cdot 3 \cdot 2 \cdot a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x + \cdots , \\ 0 &= a_4; \\ \cos x &= 5 \cdot 4 \cdot 3 \cdot 2a_5 + \cdots , \\ 1 &= 5!\, a_5, \cdots . \end{aligned} \tag{12.4}$$

We substitute these values back into (12.1) and get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{12.5}$$

You can probably see how to write more terms of this series without further computation. The $\sin x$ series converges for all x; see Example 3, Section 10.

Series obtained in this way are called *Maclaurin series* or *Taylor series about the origin*. A Taylor series in general means a series of powers of $(x-a)$, where a is some constant. It is found by writing $(x-a)$ instead of x on the right-hand side of an equation like (12.1), differentiating just as we have done, but substituting $x=a$ instead of $x=0$ at each step. Let us carry out this process in general for a function $f(x)$. As above, we assume that there is a Taylor series for $f(x)$, and write

$$\begin{aligned}
f(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots \\
&\quad + a_n(x-a)^n + \cdots , \\
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots \\
&\quad + na_n(x-a)^{n-1} + \cdots , \\
f''(x) &= 2a_2 + 3\cdot 2a_3(x-a) + 4\cdot 3a_4(x-a)^2 + \cdots \\
&\quad + n(n-1)a_n(x-a)^{n-2} + \cdots , \\
f'''(x) &= 3!\,a_3 + 4\cdot 3\cdot 2a_4(x-a) + \cdots \\
&\quad + n(n-1)(n-2)a_n(x-a)^{n-3} + \cdots , \\
&\;\;\vdots \\
f^{(n)}(x) &= n(n-1)(n-2)\cdots 1\cdot a_n + \text{terms containing powers of } (x-a).
\end{aligned} \tag{12.6}$$

[The symbol $f^{(n)}(x)$ means the nth derivative of $f(x)$.] We now put $x = a$ in each equation of (12.6) and obtain

$$\begin{aligned} &f(a) = a_0, \quad f'(a) = a_1, \quad f''(a) = 2a_2, \\ &f'''(a) = 3!\,a_3, \quad \cdots, \quad f^{(n)}(a) = n!\,a_n. \end{aligned} \tag{12.7}$$

[Remember that $f'(a)$ means to differentiate $f(x)$ and then put $x = a$; $f''(a)$ means to find $f''(x)$ and then put $x = a$, and so on.]

We can then write the Taylor series for $f(x)$ about $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots . \tag{12.8}$$

The Maclaurin series for $f(x)$ is the Taylor series about the origin. Putting $a = 0$ in (12.8), we obtain the Maclaurin series for $f(x)$:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots . \tag{12.9}$$

convergent for

$$\text{(13.1)} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad \text{all } x;$$

$$\text{(13.2)} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad \text{all } x;$$

$$\text{(13.3)} \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad \text{all } x;$$

(13.4)

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \le 1;$$

$$\text{(13.5)} \quad (1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots, \qquad |x| < 1,$$

(binomial series; p is any real number, positive or negative and $\binom{p}{n}$ is called a binomial coefficient—see method C below.)

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).