University of Anbar
College of Science
Physics Department

Techniques for Obtaining Power Series Expansion

A. Multiplying a Series by a Polynomial or by Another Series

Example 1. To find the series for $(x+1)\sin x$, we multiply $(x+1)$ times the series (13.1) and collect terms:

$$(x+1)\sin x = (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \cdots .$$

You can see that this is easier to do than taking the successive derivatives of the product $(x+1)\sin x$, and Theorem 4 assures us that the results are the same.

Example 2. To find the series for $e^x \cos x$, we multiply (13.2) by (13.3):

$$\begin{aligned}
e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
&= 1 \quad + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \\
&\qquad\quad - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!\,2!} \cdots \\
&\qquad\qquad\qquad\qquad + \frac{x^4}{4!} \cdots \\
\hline
&= 1 \quad + x + 0x^2 - \frac{x^3}{3} - \frac{x^4}{6} \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} \cdots .
\end{aligned}$$

There are two points to note here. First, as you multiply, line up the terms involving each power of x in a column; this makes it easier to combine them. Second, be careful to include *all* the terms in the product out to the power you intend to stop with, but don't include *any* higher powers. In the above example, note that we did not include the $x^3 \cdot x^2$ terms; if we wanted the x^5 term in the answer, we would have to include *all* products giving x^5 (namely, $x \cdot x^4, x^3 \cdot x^2$, and $x^5 \cdot 1$).

B. Division of Two Series or of a Series by a Polynomial

Example 1. To find the series for $(1/x)\ln(1+x)$, we divide (13.4) by x. You should be able to do this in your head and just write down the answer.

$$\frac{1}{x}\ln(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots.$$

To obtain the summation form, we again just divide (13.4) by x. We can simplify the result by changing the limits to start at $n=0$, that is, replace n by $n+1$.

$$\frac{1}{x}\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n-1}}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n+1}.$$

Example 2. To find the series for $\tan x$, we divide the series for $\sin x$ by the series for $\cos x$ by long division:

$$
\begin{array}{r|l}
 & x + \frac{x^3}{3} + \frac{2}{15}x^5 \cdots \\ \cline{2-2}
1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots & x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \\
 & x - \frac{x^3}{2!} + \frac{x^5}{4!} \cdots \\ \cline{2-2}
 & \frac{x^3}{3} - \frac{x^5}{30} \cdots \\
 & \frac{x^3}{3} - \frac{x^5}{6} \cdots \\ \cline{2-2}
 & \frac{2x^5}{15} \cdots \text{, etc.}
\end{array}
$$

C. Binomial Series

If you recall the binomial theorem, you may see that (13.5) looks just like the beginning of the binomial theorem for the expansion of $(a+b)^n$ if we put $a=1$, $b=x$, and $n=p$. The difference here is that we allow p to be negative or fractional, and in these cases the expansion is an infinite series. The series converges for $|x|<1$ as you can verify by the ratio test. (See Problem 1.)

From (13.5), we see that the binomial coefficients are:

$$\begin{aligned}
\binom{p}{0} &= 1, \\
\binom{p}{1} &= p, \\
\binom{p}{2} &= \frac{p(p-1)}{2!}, \\
\binom{p}{3} &= \frac{p(p-1)(p-2)}{3!}, \cdots, \\
\binom{p}{n} &= \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}.
\end{aligned} \tag{13.6}$$

Example 1. To find the series for $1/(1+x)$, we use the binomial series (13.5) to write

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-x)^n.$$

Example 2. The series for $\sqrt{1+x}$ is (13.5) with $p = 1/2$.

$$\begin{aligned}\sqrt{1+x} = (1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n \\ &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \cdots .\end{aligned}$$

From (13.6) we can see that the binomial coefficients when $n = 0$ and $n = 1$ are $\binom{1/2}{0} = 1$ and $\binom{1/2}{1} = 1/2$. For $n \geq 2$, we can write

$$\begin{aligned}\binom{\frac{1}{2}}{n} &= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}3\cdot 5\cdot 7\cdots(2n-3)}{n!\,2^n} \\ &= \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!}\end{aligned}$$

where the double factorial of an odd number means the product of that number times all smaller odd numbers, and a similar definition for even numbers. For example, $7!! = 7 \cdot 5 \cdot 3$, and $8!! = 8 \cdot 6 \cdot 4 \cdot 2$.

Problems

Write the Maclaurin series for $1/\sqrt{1+x}$

D. Substitution of a Polynomial or a Series for the Variable in Another Series

Example 1. Find the series for e^{-x^2}. Since we know the series (13.3) for e^x, we simply replace the x there by $-x^2$ to get

$$\begin{aligned} e^{-x^2} &= 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{(x^4)}{2!} - \frac{x^6}{3!} + \cdots . \end{aligned}$$

Example 2. Find the series for $e^{\tan x}$. Here we must replace the x in (13.3) by the series of Example 2 in method B. Let us agree in advance to keep terms only as far as x^4; we then write only terms which can give rise to powers of x up to 4, and neglect any higher powers:

$$
\begin{aligned}
e^{\tan x} &= 1 + \left(x + \frac{x^3}{3} + \cdots\right) + \frac{1}{2!}\left(x + \frac{x^3}{3} + \cdots\right)^2 \\
&\quad + \frac{1}{3!}\left(x + \frac{x^3}{3} + \cdots\right)^3 + \frac{1}{4!}(x + \cdots)^4 + \cdots \\
&= 1 + x \qquad + \frac{x^3}{3} \qquad + \cdots \\
&\quad + \frac{x^2}{2!} \qquad + \frac{2x^4}{3 \cdot 2!} + \cdots \\
&\quad + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\hline
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3}{8}x^4 + \cdots .
\end{aligned}
$$

E. Combination of Methods

Example. Find the series for $\arctan x$. Since

$$\int_0^x \frac{dt}{1+t^2} = \arctan t\Big|_0^x = \arctan x,$$

we first write out (as a binomial series) $(1+t^2)^{-1}$ and then integrate term by term:

$$(1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + \cdots;$$

$$\int_0^x \frac{dt}{1+t^2} = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots\Big|_0^x.$$

Thus, we have

(13.7) $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

F. Taylor Series Using the Basic Maclaurin Series

Example 1. Find the first few terms of the Taylor series for $\ln x$ about $x = 1$. [This means a series of powers of $(x - 1)$ rather than powers of x.] We write

$$\ln x = \ln[1 + (x - 1)]$$

and use (13.4) with x replaced by $(x - 1)$:

$$\ln x = \ln[1 + (x - 1)] = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 \cdots .$$

Example 2. Expand $\cos x$ about $x = 3\pi/2$. We write

$$\cos x = \cos\left[\frac{3\pi}{2} + \left(x - \frac{3\pi}{2}\right)\right] = \sin\left(x - \frac{3\pi}{2}\right)$$
$$= \left(x - \frac{3\pi}{2}\right) - \frac{1}{3!}\left(x - \frac{3\pi}{2}\right)^3 + \frac{1}{5!}\left(x - \frac{3\pi}{2}\right)^5 \cdots$$

using (13.1) with x replaced by $(x - 3\pi/2)$.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).