University of Anbar
College of Science
Physics Department

Introduction to Complex Numbers

INTRODUCTION

You will probably recall using imaginary and complex numbers in algebra. The general solution of the quadratic equation

$$az^2 + bz + c = 0 \tag{1.1}$$

for the unknown z, is given by the *quadratic formula*

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.2}$$

If the *discriminant* $d = (b^2 - 4ac)$ is negative, we must take the square root of a negative number in order to find z. Since only non-negative numbers have real square roots, it is impossible to use (1.2) when $d < 0$ unless we introduce a new kind of number, called an imaginary number. We use the symbol $i = \sqrt{-1}$ with the understanding that $i^2 = -1$. Then

$$\sqrt{-16} = 4i, \quad \sqrt{-3} = i\sqrt{3}, \quad i^3 = -i$$

are imaginary numbers, but

$$i^2 = -1, \quad \sqrt{-2}\sqrt{-8} = i\sqrt{2} \cdot i\sqrt{8} = -4, \quad i^{4n} = 1$$

are real. In (1.2) we also need combinations of real and imaginary numbers.

Example. The solution of

$$z^2 - 2z + 2 = 0$$

is

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

We use the term *complex number* to mean any one of the whole set of numbers, real, imaginary, or combinations of the two like $1 \pm i$. Thus, $i + 5$, $17i$, 4, $3 + i\sqrt{5}$ are all examples of complex numbers.

REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

A complex number such as $5 + 3i$ is the sum of two terms. The real term (not containing i) is called the *real part* of the complex number. The *coefficient* of i in the other term is called the *imaginary part* of the complex number. In $5 + 3i$, 5 is the real part and 3 is the imaginary part. Notice carefully that the *imaginary part* of a complex number is *not imaginary*!

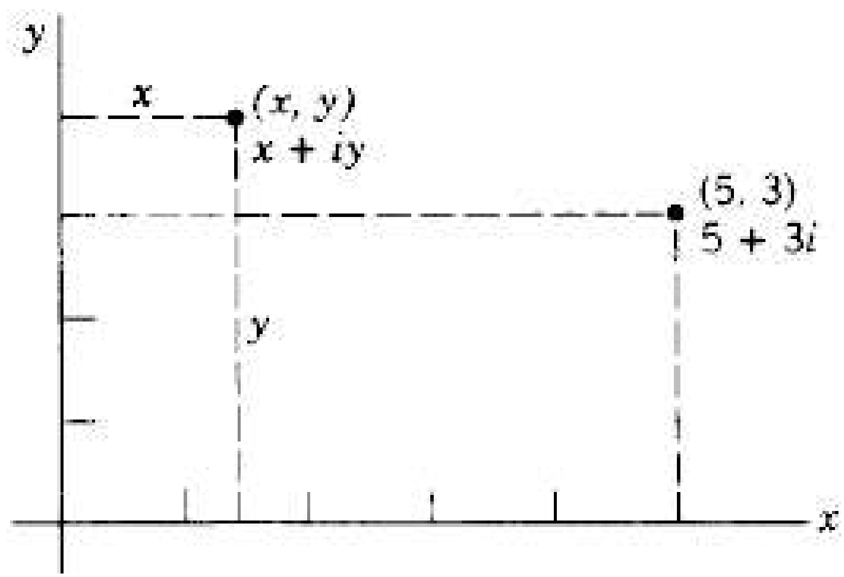
Either the real part or the imaginary part of a complex number may be zero. If the real part is zero, the complex number is called imaginary (or, for emphasis, *pure* imaginary). The zero real part is usually omitted; thus $0 + 5i$ is written just $5i$. If the imaginary part of the complex number is zero, the number is real. We write $7 + 0i$ as just 7. Complex numbers then include both real numbers and pure imaginary numbers as special cases.

In algebra a complex number is ordinarily written (as we have been doing) as a sum like $5 + 3i$. There is another very useful way of thinking of a complex number. As we have said, every complex number has a real part and an imaginary part (either of which may be zero). These are two *real* numbers, and we could, if we liked, agree to write $5+3i$ as $(5, 3)$. Any complex number could be written this way as a pair of real numbers, the real part first and then the imaginary part (which, you must remember, is real). This would not be a very convenient form for computation, but it suggests a very useful geometrical representation of a complex number which we shall now consider.

THE COMPLEX PLANE

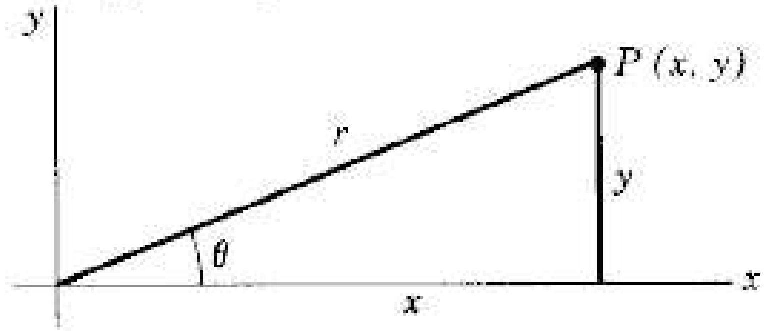
In analytic geometry we plot the point $(5, 3)$ as shown in Figure 3.1. As we have seen, the symbol $(5, 3)$ could also mean the complex number $5 + 3i$. The point $(5, 3)$ may then be labeled either $(5, 3)$ or $5 + 3i$. Similarly, any complex number $x + iy$ (x and y real) can be represented by a point (x, y) in the (x, y) plane. Also any point (x, y) in the (x, y) plane can be labeled $x + iy$ as well as (x, y). When the (x, y)

plane is used in this way to plot complex numbers, it is called the *complex plane*. It is also sometimes called an *Argand diagram*. The x axis is called the real axis, and the y axis is called the imaginary axis (note, however, that you plot y and *not* iy).



When a complex number is written in the form $x + iy$, we say that it is in *rectangular form* because x and y are the rectangular coordinates of the point representing the number in the complex plane. In analytic geometry, we can locate a point by giving its polar coordinates (r, θ) instead of its rectangular coordinates (x, y). There is a corresponding way to write any complex number.

$$x = r\cos\theta,$$
$$y = r\sin\theta. \tag{3.1}$$



Then we have

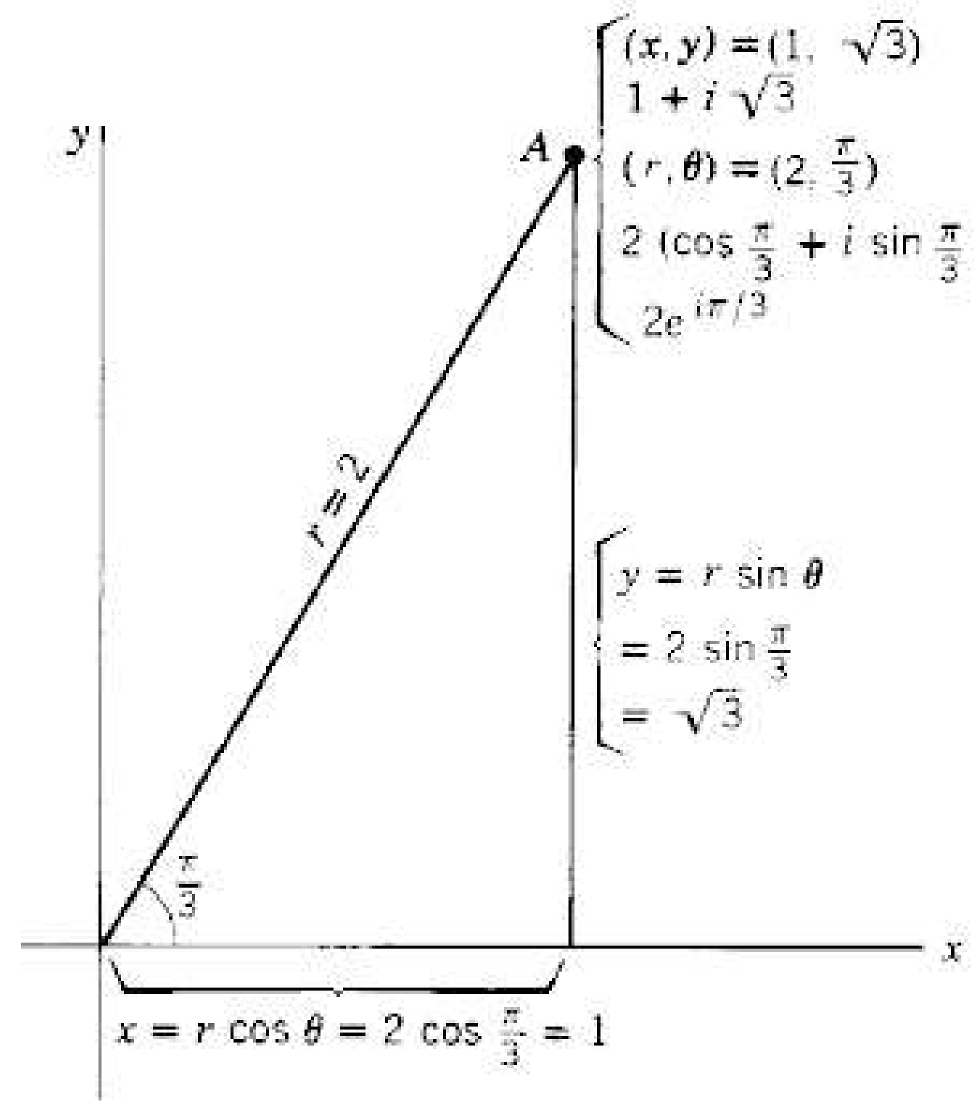
$$\begin{aligned} x + iy &= r\,\cos\theta + ir\,\sin\theta \\ &= r\,(\cos\theta + i\,\sin\theta). \end{aligned} \tag{3.2}$$

This last expression is called the *polar form* of the complex number.

$$x + iy = r(\cos\theta + i\,\sin\theta) = re^{i\theta}. \tag{3.3}$$

Example. In Figure 3.3 the point A could be labeled as $(1, \sqrt{3})$ or as $1 + i\sqrt{3}$. Similarly, using polar coordinates, the point A could be labeled with its (r, θ) values as $(2, \pi/3)$. Notice that r is always taken positive. Using (3.3) we have

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\,\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}.$$



- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).