Terminology and Notation
Formula

TERMINOLOGY AND NOTATION

Both i and j are used to represent $\sqrt{-1}$, j usually in any problem dealing with electricity since i is needed there for current. A physicist should be able to work with ease using either symbol. We shall for consistency use i throughout this book.

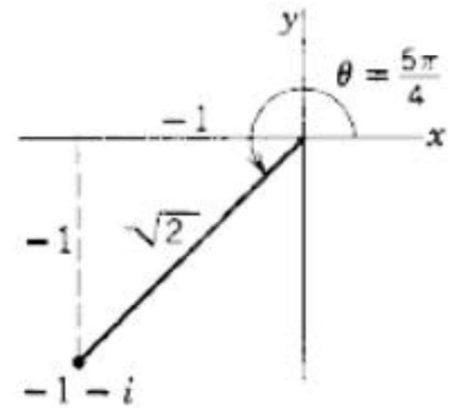
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{4.1}$$

Here z is a complex number; x is the *real part* of the complex number z, and y is the *imaginary part* of z. The quantity r is called the *modulus* or *absolute value* of z, and θ is called the *angle* of z (or the *phase*, or the *argument*, or the *amplitude* of z). In symbols:

$$\begin{aligned} &\operatorname{Re} z = x, && |z| = \operatorname{mod} z = r = \sqrt{x^2+y^2}, \\ &\operatorname{Im} z = y \,(\textbf{not } iy), && \text{angle of } z = \theta. \end{aligned} \tag{4.2}$$

Example. Write $z = -1 - i$ in polar form. Here we have $x = -1$, $y = -1$, $r = \sqrt{2}$ There are an infinite number of values of θ,

$$\theta = \frac{5\pi}{4} + 2n\pi, \tag{4.3}$$

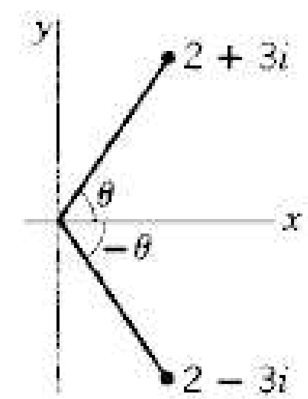


where n is any integer, positive or negative. The value $\theta = 5\pi/4$ is sometimes called the *principal angle* of the complex number $z = -1 - i$. Notice carefully, however, that this is not the same as the principal value $\pi/4$ of $\arctan 1$ as defined in calculus. The angle of a complex number must be in the same quadrant as the point representing the number. For our present work, any one of the values in (4.3) will do; here we would probably use either $5\pi/4$ or $-3\pi/4$. Then we have in our example

$$\begin{aligned} z = -1 - i &= \sqrt{2}\left[\cos\left(\frac{5\pi}{4} + 2n\pi\right) + i\sin\left(\frac{5\pi}{4} + 2n\pi\right)\right] \\ &= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\, e^{5i\pi/4}. \end{aligned}$$

The complex number $x - iy$, obtained by changing the sign of i in $z = x + iy$, is called the *complex conjugate* or simply the *conjugate* of z. We usually write the conjugate of $z = x + iy$ as $\bar{z} = x - iy$. Sometimes we use z^* instead of $\bar{z}$ (in fields such as statistics or quantum mechanics where the bar may be used to mean an average value). Notice carefully that the conjugate of $7i - 5$ is $-7i - 5$; that is, it is the i term whose sign is changed.

Complex numbers come in conjugate pairs; for example, the conjugate of $2 + 3i$ is $2 - 3i$ and the conjugate of $2-3i$ is $2+3i$. Such a pair of points in the complex plane are mirror images of each other with the x axis as the mirror (Figure 4.2). Then in polar form, z and $\bar{z}$ have the same r value, but their θ values are negatives of each other. If we write $z = r(\cos\theta + i \sin\theta)$, then



$$(4.4) \qquad \bar{z} = r[\cos(-\theta) + i \sin(-\theta)] = r(\cos\theta - i \sin\theta) = re^{-i\theta}.$$

Problems

For each of the following numbers, first visualize where it is in the complex plane. With a little practice you can quickly find x, y, r, θ in your head for these simple problems. Then

plot the number and label it in five ways as in Figure 3.3. Also plot the complex conjugate of the number.

1. $1 + i$

2. $i - 1$

3. $1 - i\sqrt{3}$

4. $-\sqrt{3} + i$

5. $2i$

6. $-4i$

7. -1

8. 3

9. $2i - 2$

10. $2 - 2i$

11. $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

12. $4\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

13. $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

14. $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

15. $\cos\pi - i\sin\pi$

16. $5(\cos 0 + i\sin 0)$

17. $\sqrt{2}\,e^{-i\pi/4}$

18. $3\,e^{i\pi/2}$

19. $5(\cos 20^\circ + i\sin 20^\circ)$

20. $7(\cos 110^\circ - i\sin 110^0)$

Example 1.

$$(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$$

To divide one complex number by another, first write the quotient as a fraction. Then reduce the fraction to rectangular form by multiplying numerator and denominator by the conjugate of the denominator; this makes the denominator real.

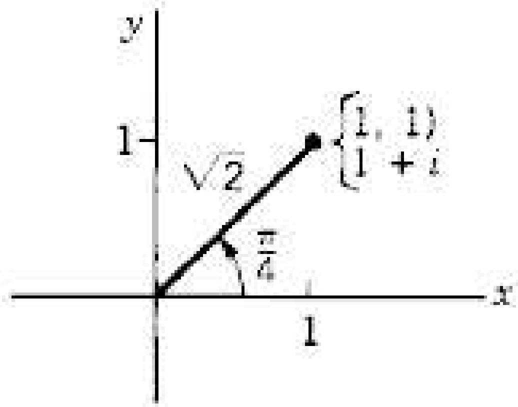
Example 2.

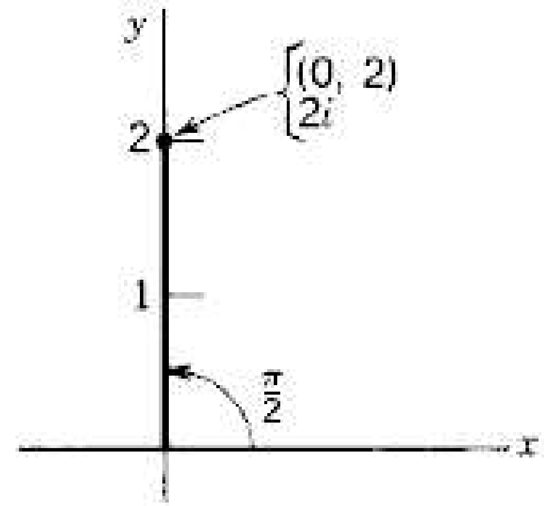
$$\frac{2+i}{3-i} = \frac{2+i}{3-i} \cdot \frac{3+i}{3+i} = \frac{6+5i+i^2}{9-i^2} = \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i.$$

It is sometimes easier to multiply or divide complex numbers in polar form.

Example 3. To find $(1+i)^2$ in polar form, we first sketch (or picture mentally) the point $(1,1)$. $r=\sqrt{2}$, and $\theta=\pi/4$, so $(1+i)=\sqrt{2}\,e^{i\pi/4}$.

$$(1+i)^2=(\sqrt{2}\,e^{i\pi/4})^2=2\,e^{i\pi/2}=2i.$$





Example 4. Write $1/[2(\cos 20° + i \sin 20°]$ in $x + iy$ form. Since $20° = \pi/9$ radians,

$$\frac{1}{2(\cos 20° + i \sin 20°)} = \frac{1}{2(\cos \pi/9 + i \sin \pi/9)} = \frac{1}{2\, e^{i\pi/9}} = 0.5\, e^{-i\pi/9}$$
$$= 0.5(\cos \pi/9 - i \sin \pi/9) = 0.47 - 0.17i,$$

by calculator in radian mode. We obtain the same result leaving the angle in degrees and using a calculator in degree mode: $0.5(\cos 20° - i \sin 20°) = 0.47 - 0.17i$.

Problems

First simplify each of the following numbers to the $x + iy$ form or to the $re^{i\theta}$ form. Then plot the number in the complex plane.

1. $\dfrac{1}{1+i}$ **2.** $\dfrac{1}{i-1}$ **3.** i^4

4. $i^2 + 2i + 1$ **5.** $\left(i + \sqrt{3}\right)^2$ **6.** $\left(\dfrac{1+i}{1-i}\right)^2$

7. $\dfrac{3+i}{2+i}$ **8.** $1.6 - 2.7i$

9. $25\,e^{2i}$ *Careful!* The angle is 2 radians.

10. $\dfrac{3i-7}{i+4}$ *Careful! Not* $3 - 7i$

11. $17 - 12i$ **12.** $3(\cos 28^\circ + i \sin 28^\circ)$

13. $5\left(\cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}\right)$ **14.** $2.8e^{-i(1.1)}$

15. $\dfrac{5-2i}{5+2i}$ **16.** $\dfrac{1}{0.5(\cos 40^\circ + i \sin 40^\circ)}$

17. $(1.7 - 3.2i)^2$

18. $(0.64 + 0.77i)^4$

Find each of the following in rectangular $(a + bi)$ form if $z = 2 - 3i$; if $z = x + iy$.

19. z^{-1}

20. $\dfrac{1}{z^2}$

21. $\dfrac{1}{z+1}$

22. $\dfrac{1}{z-i}$

23. $\dfrac{1+z}{1-z}$

24. $z/\bar{z}$

Complex Conjugate of a Complex Expression

It is easy to see that the conjugate of the sum of two complex numbers is the sum of the conjugates of the numbers. If

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2,$$

then

$$\bar{z}_1 + \bar{z}_2 = x_1 - iy_1 + x_2 - iy_2 = x_1 + x_2 - i(y_1 + y_2).$$

The conjugate of $(z_1 + z_2)$ is

$$\overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2).$$

Example. If

$$z = \frac{2-3i}{i+4}, \qquad \text{then} \qquad \bar{z} = \frac{2+3i}{-i+4}.$$

But if $z = f + ig$, where f and g are themselves complex, then the complex conjugate of z is $\bar{z} = \bar{f} - i\bar{g}$ (not $f - ig$).

Finding the Absolute Value of z

Recall that the definition of $|z|$ is $|z| = r = \sqrt{x^2+y^2}$ (positive square root!). Since $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$, or, in polar coordinates, $z\bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2$, we see that $|z|^2 = z\bar{z}$, or $|z| = \sqrt{z\bar{z}}$. Note that $z\bar{z}$ is always real and ≥ 0, since x, y, and r are real. We have

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}. \tag{5.1}$$

Example.

$$\left|\frac{\sqrt{5}+3i}{1-i}\right| = \frac{|\sqrt{5}+3i|}{|1-i|} = \frac{\sqrt{14}}{\sqrt{2}} = \sqrt{7}.$$

Problems

Find the absolute value of each of the following using the discussion above. Try to do simple problems like these in your head—it saves time.

26. $\dfrac{2i-1}{i-2}$

27. $\dfrac{2+3i}{1-i}$

28. $\dfrac{z}{\bar{z}}$

29. $(1+2i)^3$

30. $\dfrac{3i}{i-\sqrt{3}}$

31. $\dfrac{5-2i}{5+2i}$

32. $(2-3i)^4$

33. $\dfrac{25}{3+4i}$

34. $\left(\dfrac{1+i}{1-i}\right)^5$

Complex Equations

Example. Find x and y if

$$(x+iy)^2 = 2i. \tag{5.2}$$

Since $(x+iy)^2 = x^2 + 2ixy - y^2$, (5.2) is equivalent to the two real equations

$$\begin{aligned} x^2 - y^2 &= 0, \\ 2xy &= 2. \end{aligned}$$

From the first equation $y^2 = x^2$, we find $y = x$ or $y = -x$. Substituting these into the second equation gives

$$2x^2 = 2 \qquad \text{or} \qquad -2x^2 = 2.$$

Since x is real, x^2 cannot be negative. Thus we find only

$$x^2 = 1 \qquad \text{and} \qquad y = x,$$

that is,

$$x = y = 1 \qquad \text{and} \qquad x = y = -1.$$

Problems

Solve for all possible values of the real numbers x and y in the following equations.

35. $x + iy = 3i - 4$

36. $2ix + 3 = y - i$

37. $x + iy = 0$

38. $x + iy = 2i - 7$

39. $x + iy = y + ix$

40. $x + iy = 3i - ix$

41. $(2x - 3y - 5) + i(x + 2y + 1) = 0$

42. $(x + 2y + 3) + i(3x - y - 1) = 0$

43. $(x + iy)^2 = 2ix$

44. $x + iy = (1 - i)^2$

45. $(x + iy)^2 = (x - iy)^2$

46. $\dfrac{x + iy}{x - iy} = -i$

47. $(x + iy)^3 = -1$

48. $\dfrac{x + iy + 2 + 3i}{2x + 2iy - 3} = i + 2$

49. $|1 - (x + iy)| = x + iy$

50. $|x + iy| = y - ix$

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).