Euler's Formula

EULER'S FORMULA

For real θ, we know from Chapter 1 the power series for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots, \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots. \end{aligned} \tag{9.1}$$

From our definition (8.1), we can write the series for e to any power, real or imaginary. We write the series for $e^{i\theta}$, where θ is real:

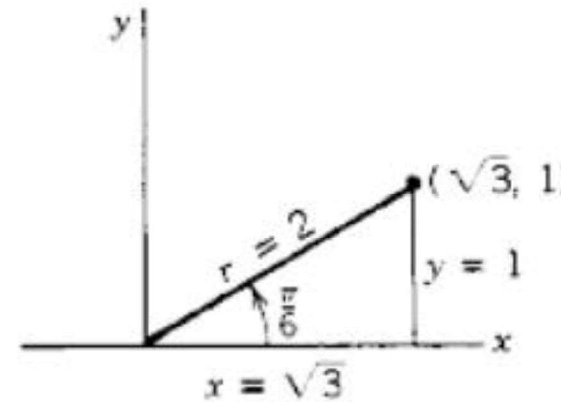
$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right). \end{aligned} \tag{9.2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9.3}$$

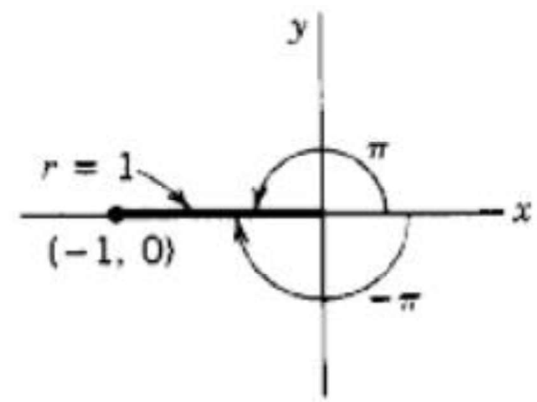
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{9.4}$$

Examples. Find the values of $2\,e^{i\pi/6}$, $e^{i\pi}$, $3\,e^{-i\pi/2}$, $e^{2n\pi i}$.

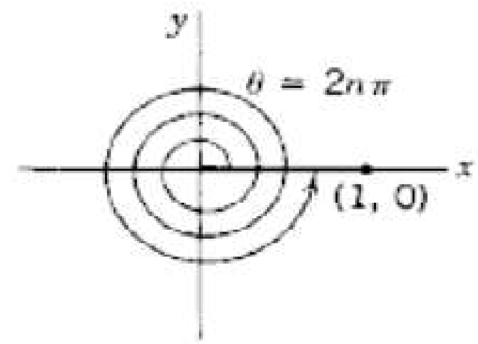
$2\,e^{i\pi/6}$ is $re^{i\theta}$ with $r = 2$, $\theta = \pi/6$. From Figure 9.1, $x = \sqrt{3}$, $y = 1$, $x + iy = \sqrt{3} + i$, so $2\,e^{i\pi/6} = \sqrt{3} + i$.



$e^{i\pi}$ is $re^{i\theta}$ with $r = 1$, $\theta = \pi$. From Figure 9.2, $x = -1$, $y = 0$, $x + iy = -1 + 0i$, so $e^{i\pi} = -1$. Note that $r = 1$ and $\theta = -\pi, \pm 3\pi, \pm 5\pi, \cdots$, give the same point, so $e^{-i\pi} = -1$, $e^{3\pi i} = -1$, and so on.



$e^{2n\pi i}$ is $re^{i\theta}$ with $r = 1$ and $\theta = 2n\pi = n(2\pi)$; that is, θ is an integral multiple of 2π. $x = 1$, $y = 0$, so $e^{2n\pi i} = 1 + 0i = 1$.



It is often convenient to use Euler's formula when we want to multiply or divide complex numbers. From (8.2) we obtain two familiar looking laws of exponents which are now valid for imaginary exponents:

$$\begin{aligned} e^{i\theta_1} \cdot e^{i\theta_2} &= e^{i(\theta_1+\theta_2)}, \\ e^{i\theta_1} \div e^{i\theta_2} &= e^{i(\theta_1-\theta_2)}. \end{aligned} \tag{9.5}$$

Remembering that *any* complex number can be written in the form $re^{i\theta}$ by (9.4), we get

$$
z_1 \cdot z_2 = r_1\, e^{i\theta_1} \cdot r_2\, e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1+\theta_2)},
$$

$$
z_1 \div z_2 = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}. \tag{9.6}
$$

In words, to multiply two complex numbers, we multiply their absolute values and add their angles. To divide two complex numbers, we divide the absolute values and subtract the angles.

POWERS AND ROOTS OF COMPLEX NUMBERS

Using the rules (9.6) for multiplication and division of complex numbers, we have

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \tag{10.1}$$

for any integral n. In words, to obtain the nth power of a complex number, we take the nth power of the modulus and multiply the angle by n. The case $r = 1$ is of particular interest. Then (10.1) becomes DeMoivre's theorem:

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{10.2}$$

You can use this equation to find the formulas for $\sin 2\theta$, $\cos 2\theta$, $\sin 3\theta$, etc. (Problems 27 and 28).

The nth root of z, $z^{1/n}$, means a complex number whose nth power is z. From (10.1) you can see that this is

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right). \tag{10.3}$$

Example 1.

$$[\cos(\pi/10) + i\sin(\pi/10)]^{25} = (e^{i\pi/10})^{25} = e^{2\pi i}e^{i\pi/2} = 1 \cdot i = i.$$

THE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Although we have already defined e^z by a power series (8.1), it is worth while to write it in another form. By (8.2) we can write

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \tag{11.1}$$

This is more convenient to use than the infinite series if we want values of e^z for given z. For example,

$$e^{2-i\pi} = e^2 e^{-i\pi} = e^2 \cdot (-1) = -e^2$$

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$$e^{i\theta} = \cos\theta + i\sin\theta,$$
$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{11.2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{11.3}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{11.4}$$

Example 1. $\cos i = \dfrac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \dfrac{e^{-1} + e}{2} = 1.543\cdots.$

Example 2.

$$\sin\left(\frac{\pi}{2} + i\ln 2\right) = \frac{e^{i(\pi/2+i\ln 2)} - e^{-i(\pi/2+i\ln 2)}}{2i}$$
$$= \frac{e^{i\pi/2}e^{-\ln 2} - e^{-i\pi/2}e^{\ln 2}}{2i} \qquad \text{by (8.2).}$$

From Figures 5.2 and 9.3, $e^{i\pi/2} = i$, and $e^{-i\pi/2} = -i$. By the definition of $\ln x$ [or see equations (13.1) and (13.2)], $e^{\ln 2} = 2$, so $e^{-\ln 2} = 1/e^{\ln 2} = 1/2$. Then

$$\sin\left(\frac{\pi}{2} + i\ln 2\right) = \frac{(i)(1/2) - (-i)(2)}{2i} = \frac{5}{4}.$$

Example 3. Prove that $\sin^2 z + \cos^2 z = 1$.

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} - 2 + e^{-2iz}}{-4},$$

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + 2 + e^{-2iz}}{4},$$

$$\sin^2 z + \cos^2 z = \frac{2}{4} + \frac{2}{4} = 1.$$

Example 4. Using the definitions (11.4), verify that $(d/dz)\sin z = \cos z$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

$$\frac{d}{dz}\sin z = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

HYPERBOLIC FUNCTIONS

Let us look at $\sin z$ and $\cos z$ for pure imaginary z, that is, $z = iy$:

$$\begin{aligned} \sin iy &= \frac{e^{-y} - e^{y}}{2i} = i\,\frac{e^{y} - e^{-y}}{2}, \\ \cos iy &= \frac{e^{-y} + e^{y}}{2} = \frac{e^{y} + e^{-y}}{2}. \end{aligned} \tag{12.1}$$

The real functions on the right have special names because these particular combinations of exponentials arise frequently in problems. They are called the hyperbolic sine (abbreviated sinh) and the hyperbolic cosine (abbreviated cosh). Their definitions for all z are

$$\begin{aligned} \sinh z &= \frac{e^{z} - e^{-z}}{2}, \\ \cosh z &= \frac{e^{z} + e^{-z}}{2}. \end{aligned} \tag{12.2}$$

The other hyperbolic functions are named and defined in a similar way to parallel the trigonometric functions:

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, & \coth z &= \frac{1}{\tanh z}, \\ \operatorname{sech} z &= \frac{1}{\cosh z}, & \operatorname{csch} z &= \frac{1}{\sinh z}. \end{aligned} \tag{12.3}$$

(See Problem 38 for the reason behind the term "hyperbolic" functions.)

We can write (12.1) as

$$\begin{aligned} \sin iy &= i \sinh y, \\ \cos iy &= \cosh y. \end{aligned} \tag{12.4}$$

Example. You can prove the following formulas

$$\cosh^2 z - \sinh^2 z = 1 \qquad \text{(compare } \sin^2 z + \cos^2 z = 1\text{)},$$

$$\frac{d}{dz}\cosh z = \sinh z \qquad \text{(compare } \frac{d}{dz}\cos z = -\sin z\text{)}.$$

Problems

Verify each of the following

1. $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

2. $\cos z = \cos x \cosh y - i \sin x \sinh y$

3. $\sinh z = \sinh x \cos y + i \cosh x \sin y$

4. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

5. $\sin 2z = 2 \sin z \cos z$

6. $\cos 2z = \cos^2 z - \sin^2 z$

7. $\sinh 2z = 2 \sinh z \cosh z$

8. $\cosh 2z = \cosh^2 z + \sinh^2 z$

9. $\dfrac{d}{dz} \cos z = -\sin z$

10. $\dfrac{d}{dz} \cosh z = \sinh z$

11. $\cosh^2 z - \sinh^2 z = 1$

12. $\cos^4 z + \sin^4 z = 1 - \dfrac{1}{2} \sin^2 2z$

13. $\cos 3z = 4 \cos^3 z - 3 \cos z$

14. $\sin iz = i \sinh z$

15. $\sinh iz = i \sin z$

16. $\tan iz = i \tanh z$

17. $\tanh iz = i \tan z$

LOGARITHMS

In elementary mathematics you learned to find logarithms of positive numbers only; in fact, you may have been told that there were no logarithms of negative numbers. This is true if you use only real numbers, but it is not true when we allow complex numbers as answers. We shall now see how to find the logarithm of any complex number $z \neq 0$ (including negative real numbers as a special case). If

$$z = e^w, \tag{13.1}$$

then by definition

$$w = \ln z. \tag{13.2}$$

(We use ln for natural logarithms to avoid the cumbersome $\log_e$ and to avoid confusion with logarithms to the base 10.)

We can write the law of exponents (8.2), using the letters of (13.1), as

$$z_1 z_2 = e^{w_1} \cdot e^{w_2} = e^{w_1 + w_2}. \tag{13.3}$$

Taking logarithms of this equation, that is, using (13.1) and (13.2), we get

$$\ln z_1 z_2 = w_1 + w_2 = \ln z_1 + \ln z_2. \tag{13.4}$$

This is the familiar law for the logarithm of a product, justified now for complex numbers. We can then find the real and imaginary parts of the logarithm of a complex number from the equation

$$w = \ln z = \ln(re^{i\theta}) = \operatorname{Ln} r + \ln e^{i\theta} = \operatorname{Ln} r + i\theta, \tag{13.5}$$

where $\operatorname{Ln} r$ means the ordinary real logarithm to the base e of the real positive number r.

Since θ has an infinite number of values (all differing by multiples of 2π), a complex number has infinitely many logarithms, differing from each other by multiples of $2\pi i$. The *principal value* of $\ln z$ (often written as $\operatorname{Ln} z$) is the one using the principal value of θ, that is $0 \leq \theta < 2\pi$. (Some references use $-\pi < \theta \leq \pi$.)

Example 1. Find $\ln(-1)$. From Figure 9.2, we see that the polar coordinates of the point $z = -1$ are $r = 1$ and $\theta = \pi, -\pi, 3\pi, \cdots$. Then,

$$\ln(-1) = \mathrm{Ln}(1) + i(\pi \pm 2n\pi) = i\pi, -i\pi, 3\pi i, \cdots .$$

Example 2. Find $\ln(1+i)$. From Figure 5.1, for $z = 1 + i$, we find $r = \sqrt{2}$, and $\theta = \pi/4 \pm 2n\pi$. Then

$$\ln(1+i) = \mathrm{Ln}\sqrt{2} + i\left(\frac{\pi}{4} \pm 2n\pi\right) = 0.347\cdots + i\left(\frac{\pi}{4} \pm 2n\pi\right).$$

Even a positive real number now has infinitely many logarithms, since its angle can be taken as 0, 2π, -2π, etc. Only one of these logarithms is real, namely the principal value $\mathrm{Ln}\, r$ using the angle $\theta = 0$.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).