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Partial Differentiation

INTRODUCTION AND NOTATION

If $y = f(x)$, then dy/dx can be thought of either as the slope of the curve $y = f(x)$ or as the rate of change of y with respect to x. Rates occur frequently in physics; time rates such as velocity, acceleration, and rate of cooling of a hot body are obvious examples. There are also other rates: rate of change of volume of a gas with applied pressure, rate of decrease of the fuel in your automobile tank with distance traveled, and so on. Equations involving rates (differential equations) often need to be solved in applied problems. Derivatives are also used in finding maximum and minimum points of a curve and in finding the power series of a function. All these applications, and more, occur also when we consider a function of several variables.

Let z be a function of two variables x and y; we write $z = f(x, y)$. Just as we think of $y = f(x)$ as a curve in two dimensions, so it is useful to interpret $z = f(x, y)$ geometrically. If x, y, z are rectangular coordinates, then for each x, y the equation gives us a value of z, and so determines a point (x, y, z) in three dimensions. All the points satisfying the equation ordinarily form a surface in three-dimensional space (see Figure 1.1). (It might happen that an equation would not be satisfied by any real points, for example $x^2 + y^2 + z^2 = -1$, but we shall be interested in equations whose graphs are real surfaces.) Now suppose x is constant; think of a plane $x =$ const. intersecting the surface (see Figure 1.1). The points satisfying $z = f(x, y)$ *and* $x =$ const. then lie on a curve (the curve of intersection of the surface and the $x =$ const. plane; this is AB in Figure 1.1). We might want the slope, maximum and minimum points, etc., of this curve. Since z is a function of y (on this curve), we might write dz/dy for the slope. However, to show that z is actually a function of two variables x and y with one of them (x) temporarily a constant, we write $\partial z/\partial y$; we call $\partial z/\partial y$ the partial derivative of z with respect to y. Similarly, we can

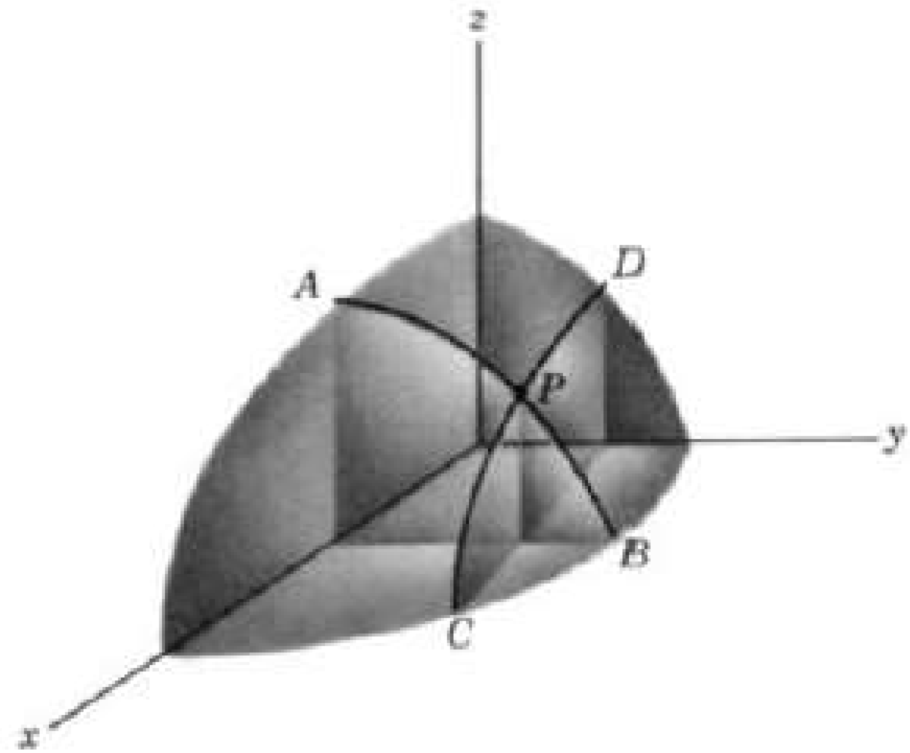

Figure 1.1

hold y constant and find $\partial z/\partial x$, the partial derivative of z with respect to x. If these partial derivatives are differentiated further, we write

$$\frac{\partial}{\partial x}\frac{\partial z}{\partial x}=\frac{\partial^2 z}{\partial x^2},\qquad \frac{\partial}{\partial x}\frac{\partial z}{\partial y}=\frac{\partial^2 z}{\partial x\partial y},\qquad \frac{\partial}{\partial x}\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^3 z}{\partial x^2\partial y},\qquad \text{etc.}$$

Other notations are often useful. If $z=f(x,y)$, we may use z_x or f_x or f_1 for $\partial f/\partial x$, and corresponding notations for the higher derivatives.

Example. Given $z = f(x, y) = x^3 y - e^{xy}$, then

$$\frac{\partial f}{\partial x} \equiv \frac{\partial z}{\partial x} \equiv f_x \equiv z_x \equiv f_1 = 3x^2 y - ye^{xy},$$
$$\frac{\partial f}{\partial y} \equiv \frac{\partial z}{\partial y} \equiv f_y \equiv z_y \equiv f_2 = x^3 - xe^{xy},$$
$$\frac{\partial^2 f}{\partial x\,\partial y} \equiv \frac{\partial^2 z}{\partial x\,\partial y} \equiv f_{yx} \equiv z_{yx} \equiv f_{21} = 3x^2 - e^{xy} - xye^{xy},$$
$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial^2 z}{\partial x^2} \equiv f_{xx} \equiv z_{xx} \equiv f_{11} = 6xy - y^2 e^{xy},$$
$$\frac{\partial^3 f}{\partial y^3} \equiv \frac{\partial^3 z}{\partial y^3} \equiv f_{yyy} \equiv z_{yyy} \equiv f_{222} = -x^3 e^{xy},$$
$$\frac{\partial^3 f}{\partial x^2\,\partial y} \equiv \frac{\partial^3 z}{\partial x^2\,\partial y} \equiv f_{yxx} \equiv z_{yxx} \equiv f_{211} = 6x - 2ye^{xy} - xy^2 e^{xy}.$$

We can also consider functions of more variables than two, although in this case it is not so easy to give a geometrical interpretation. For example, the temperature T of the air in a room might depend on the point (x, y, z) at which we measured it and on the time t; we would write $T = T(x, y, z, t)$. We could then find, say, $\partial T/\partial y$, meaning the rate at which T is changing with y for fixed x and z at one instant of time t.

A notation which is frequently used in applications (particularly thermodynamics) is $(\partial z/\partial x)_y$, meaning $\partial z/\partial x$ when z is expressed as a function of x and y. (Note two different uses of the subscript y; in the example above, f_y meant $\partial f/\partial y$. A subscript *on a partial derivative*, however, does *not* mean another derivative, but just indicates the variable being held constant in the indicated partial differentiation.) For example, let $z = x^2 - y^2$. Then using polar coordinates r and θ, (recall that $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$), we can write z in several other ways. For each new expression let us find $\partial z/\partial r$.

$$\begin{aligned}
&z = x^2 - y^2, \\
&z = r^2\cos^2\theta - r^2\sin^2\theta, && \left(\frac{\partial z}{\partial r}\right)_\theta = 2r(\cos^2\theta - \sin^2\theta), \\
&z = 2x^2 - x^2 - y^2 = 2x^2 - r^2, && \left(\frac{\partial z}{\partial r}\right)_x = -2r, \\
&z = x^2 + y^2 - 2y^2 = r^2 - 2y^2, && \left(\frac{\partial z}{\partial r}\right)_y = +2r.
\end{aligned}$$

These three expressions for $\partial z/\partial r$ have different values and are derivatives of three different functions, so we distinguish them as indicated by writing the second independent variable as a subscript. Note that we do *not* write $z(x, y)$ or $z(r, \theta)$; z is *one* variable, but it is equal to several *different* functions. Pure mathematics books usually avoid the subscript notation by writing, say, $z = f(r, \theta) = g(r, x) = h(r, y)$, etc.; then $(\partial z/\partial r)_\theta$ can be written as just $\partial f/\partial r$, and similarly

$$\left(\frac{\partial z}{\partial r}\right)_x = \frac{\partial g}{\partial r} \qquad \text{and} \qquad \left(\frac{\partial z}{\partial r}\right)_y = \frac{\partial h}{\partial r}.$$

However, this multiplicity of notation ($z = f = g = h$, etc.) would be inconvenient and confusing in applications where the letters have *physical* meanings. For example, in thermodynamics, we might need

$$\left(\frac{\partial T}{\partial p}\right)_v, \qquad \left(\frac{\partial T}{\partial v}\right)_s, \qquad \left(\frac{\partial T}{\partial p}\right)_u, \qquad \left(\frac{\partial T}{\partial s}\right)_p, \qquad \text{etc.,}$$

as well as many other similar partial derivatives. Now T means temperature (and the other letters similarly have physical meanings which must be recognized). If we wrote $T = A(p, v) = B(v, s) = C(p, u) = D(s, p)$ and similar formulas for the eight commonly used quantities in thermodynamics, each as functions of pairs from the other seven, we would not only have an unwieldy system, but the physical meaning of equations would be lost until we translated them back to standard letters. Thus the subscript notation is essential.

The symbol $(\partial z/\partial r)_x$ is usually read "the partial of z with respect to r, with x held constant." However, the important point to understand is that the notation means that z has been written as a function of the variables r and x *only*, and then differentiated with respect to r.

A little experimenting with various functions $f(x, y)$ will probably convince you that $(\partial/\partial x)(\partial f/\partial y) = (\partial/\partial y)(\partial f/\partial x)$; this is usually (but not always) true in applied problems. It can be proved (see advanced calculus texts) that if the first and second order partial derivatives of f are continuous, then $\partial^2 f/\partial x\,\partial y$ and $\partial^2 f/\partial y\,\partial x$ *are* equal. In many applied problems, these conditions are met; for example, in thermodynamics they are normally assumed and are called the reciprocity relations.

Problems

1. If $u = x^2/(x^2 + y^2)$, find $\partial u/\partial x$, $\partial u/\partial y$.

2. If $s = t^u$, find $\partial s/\partial t$, $\partial s/\partial u$.

3. If $z = \ln\sqrt{u^2 + v^2 + w^2}$, find $\partial z/\partial u$, $\partial z/\partial v$, $\partial z/\partial w$.

4. For $w = x^3 - y^3 - 2xy + 6$, find $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ at the points where $\partial w/\partial x = \partial w/\partial y = 0$.

5. For $w = 8x^4 + y^4 - 2xy^2$, find $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ at the points where $\partial w/\partial x = \partial w/\partial y = 0$.

6. For $u = e^x \cos y$,

(a) verify that $\partial^2 u/\partial x\partial y = \partial^2 u/\partial y\partial x$;

(b) verify that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).

Power Series in Two Variables

POWER SERIES IN TWO VARIABLES

Example 1. Expand $f(x, y) = \sin x \cos y$ in a two-variable Maclaurin series. We write and multiply the series for $\sin x$ and $\cos y$. This gives

$$\sin x \cos y = \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{y^2}{2!} + \cdots\right) = x - \frac{x^3}{3!} - \frac{xy^2}{2!} + \cdots .$$

Example 2. Find the two-variable Maclaurin series for $\ln(1+x-y)$. We replace x in equation (13.4) of Chapter 1 by $x-y$ to get

$$\begin{aligned}\ln(1+x-y) &= (x-y)-(x-y)^2/2+(x-y)^3/3+\cdots\\ &= x-y-x^2/2+xy-y^2/2+x^3/3-x^2y+xy^2-y^3/3+\cdots.\end{aligned}$$

The methods of Chapter 1, used as we have just shown, provide an easy way of obtaining the power series for many simple functions $f(x,y)$. However, it is also convenient, for theoretical purposes, to have formulas for the coefficients in the Taylor series or the Maclaurin series for $f(x,y)$; see, for example, Problem 8.2. Following a process similar to that used in Chapter 1, Section 12, we can find the coefficients of the power series for a function of two variables $f(x,y)$ (assuming that it can be expanded in a power series). To find the series expansion of $f(x,y)$ about the point (a,b) we write $f(x,y)$ as a series of powers of $(x-a)$ and $(y-b)$ and then differentiate this equation repeatedly as follows.

$$\begin{aligned}f(x,y) &= a_{00}+a_{10}(x-a)+a_{01}(y-b)+a_{20}(x-a)^2+a_{11}(x-a)(y-b)\\ &\quad+a_{02}(y-b)^2+a_{30}(x-a)^3+a_{21}(x-a)^2(y-b)\\ &\quad+a_{12}(x-a)(y-b)^2+a_{03}(y-b)^3+\cdots.\\ f_x &= a_{10}+2a_{20}(x-a)+a_{11}(y-b)+\cdots,\\ f_y &= a_{01}+a_{11}(x-a)+2a_{02}(y-b)+\cdots,\\ f_{xx} &= 2a_{20}+\text{ terms containing }(x-a)\text{ and/or }(y-b),\\ f_{xy} &= a_{11}+\text{ terms containing }(x-a)\text{ and/or }(y-b).\end{aligned} \tag{2.1}$$

[We have written only a few derivatives to show the idea. You should be able to calculate others in the same way (Problem 7).] Now putting $x=a$, $y=b$ in (2.1), we get

$$f(a,b) = a_{00}, \quad f_x(a,b) = a_{10}, \quad f_y(a,b) = a_{01},$$
$$f_{xx}(a,b) = 2a_{20}, \quad f_{xy}(a,b) = a_{11}, \quad \text{etc.} \tag{2.2}$$

[Remember that $f_x(a,b)$ means that we are to find the partial derivative of f with respect to x and then put $x = a$, $y = b$, and similarly for the other derivatives.] Substituting the values for the coefficients into (2.1), we find

(2.3)
$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2!}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2] \cdots .$$

This can be written in a simpler form if we put $x - a = h$ and $y - b = k$. Then the second-order terms (for example) become

$$\frac{1}{2!}[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2]. \tag{2.4}$$

We can write this in the form

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) \tag{2.5}$$

if we understand that the parenthesis is to be squared and then a term of the form $h(\partial/\partial x)k(\partial/\partial y)f(a,b)$ is to mean $hkf_{xy}(a,b)$. It can be shown (Problem 7) that the third-order terms can be written in this notation as

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(a,b)=\frac{1}{3!}[h^3 f_{xxx}(a,b)+3h^2kf_{xxy}(a,b)+\cdots] \tag{2.6}$$

and so on for terms of any order. Thus we can write the series (2.3) in the form

$$f(x,y)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a,b). \tag{2.7}$$

Problems

Find the two-variable Maclaurin series for the following functions.

1. $\cos x \sinh y$ **2.** $\cos(x+y)$ **3.** $\dfrac{\ln(1+x)}{1+y}$

4. e^{xy} **5.** $\sqrt{1+xy}$ **6.** e^{x+y}

7. Verify the coefficients of the third-order terms [(2.6) or $n = 3$ in (2.7)] of the power series for $f(x, y)$ by finding the third-order partial derivatives in (2.1) and substituting $x = a$, $y = b$.

8. Find the two-variable Maclaurin series for $e^x \cos y$ and $e^x \sin y$ by finding the series for $e^z = e^{x+iy}$ and taking real and imaginary parts.

TOTAL DIFFERENTIALS

The graph (Figure 3.1) of the equation $y = f(x)$ is a curve in the (x, y) plane and

$$y' = \frac{dy}{dx} = \frac{d}{dx} f(x) \tag{3.1}$$

is the slope of the tangent to the curve at the point (x, y). In calculus, we use Δx to mean a change in x, and Δy means the corresponding change in y (see Figure 3.1). By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2}$$

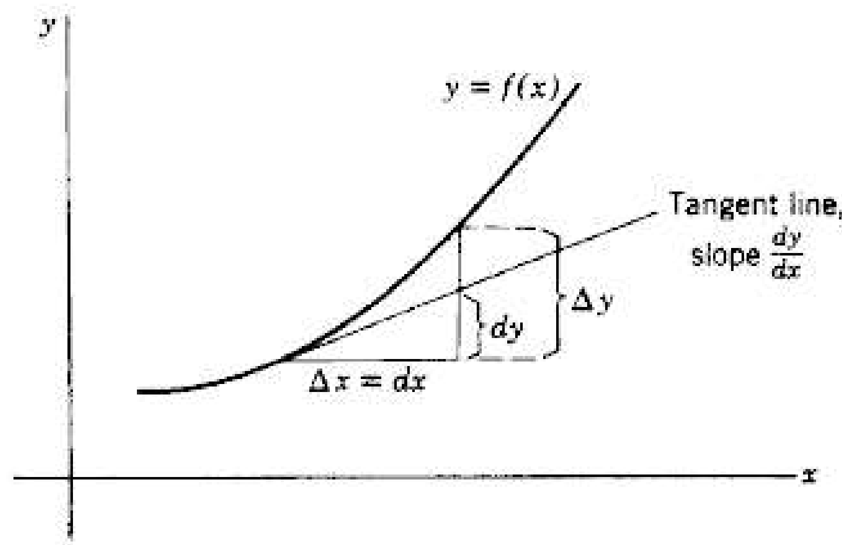

Figure 3.1

We shall now define the differential dx of the independent variable as

$$dx = \Delta x. \tag{3.3}$$

However, dy is not the same as Δy. From Figure 3.1 and equation (3.1), we can see that Δy is the change in y along the curve, but $dy = y'dx$ is the change in y along the tangent line. We say that dy is the tangent approximation (or linear approximation) to Δy.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).