Differentiating Function of a Function

Example 1. Find dy/dx if $y = \ln \sin 2x$.

You would say

$$\frac{dy}{dx} = \frac{1}{\sin 2x} \cdot \frac{d}{dx}(\sin 2x) = \frac{1}{\sin 2x} \cdot \cos 2x \cdot \frac{d}{dx}(2x) = 2 \cot 2x.$$

We *could* write this problem as

$$y = \ln u, \qquad \text{where} \qquad u = \sin v \qquad \text{and} \qquad v = 2x.$$

Then we would say

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

This is an example of the chain rule. We shall want a similar equation for a function of several variables. Consider another example.

Example 2. Find dz/dt if $z = 2t^2 \sin t$.

Differentiating the product, we get

$$\frac{dz}{dt} = 4t \sin t + 2t^2 \cos t.$$

We *could* have written this problem as

$$z = xy, \qquad \text{where } \ x = 2t^2 \qquad \text{and } \ y = \sin t,$$
$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}.$$

But since x is $\partial z/\partial y$ and y is $\partial z/\partial x$, we could also write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{5.1}$$

We would like to be sure that (5.1) is a correct formula in general, when we are given any function $z(x, y)$ with continuous partial derivatives and x and y are differentiable functions of t. To see this, recall from our discussion of differentials that we had

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{5.2}$$

where ϵ_1 and $\epsilon_2 \to 0$ with Δx and Δy. Divide this equation by Δt and let $\Delta t \to 0$; since Δx and $\Delta y \to 0$, ϵ_1 and $\epsilon_2 \to 0$ also, and we get (5.1).

It is often convenient to use differentials rather than derivatives as in (5.1). We would like to be able to use (3.6), but in (3.6) x and y were independent variables and now they are functions of t. However, it is possible to show (Problem 8) that dz as defined in (3.6) is a good approximation to Δz even though x and y are related. We may then write

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{5.3}$$

whether or not x and y are independent variables, and we may think of getting (5.1) by dividing (5.3) by dt. This is very convenient in doing problems. Thus we could do Example 2 in the following way:

$$dz = x\,dy + y\,dx = x\cos t\,dt + y \cdot 4t\,dt = (2t^2\cos t + 4t\sin t)dt,$$

$$\frac{dz}{dt} = 2t^2\cos t + 4t\sin t.$$

Example 3. Find dz/dt given $z = x^y$, where $y = \tan^{-1} t$, $x = \sin t$.
Using differentials, we find

$$dz = yx^{y-1}\,dx + x^y \ln x\,dy = yx^{y-1}\cos t\,dt + x^y \ln x \cdot \frac{dt}{1+t^2},$$

$$\frac{dz}{dt} = yx^{y-1}\cos t + x^y \ln x \cdot \frac{1}{1+t^2}.$$

Problems

1. Given $z = xe^{-y}$, $x = \cosh t$, $y = \cos t$, find dz/dt.

2. Given $w = \sqrt{u^2 + v^2}$, $u = \cos[\ln \tan(p + \frac{1}{4}\pi)]$, $v = \sin[\ln \tan(p + \frac{1}{4}\pi)]$, find dw/dp.

3. Given $r = e^{-p^2 - q^2}$, $p = e^s$, $q = e^{-s}$, find dr/ds.

4. Given $x = \ln(u^2 - v^2)$, $u = t^2$, $v = \cos t$, find dx/dt.

5. If we are given $z = z(x, y)$ and $y = y(x)$, show that the chain rule (5.1) gives

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

6. Given $z = (x + y)^5$, $y = \sin 10x$, find dz/dx.

7. Given $c = \sin(a - b)$, $b = ae^{2a}$, find dc/da.

IMPLICIT DIFFERENTIATION

Example 1. Given $x + e^x = t$, find dx/dt and d^2x/dt^2.

If we give values to x, find the corresponding t values, and plot x against t, we have a graph whose slope is dx/dt. In other words, x *is* a function of t even though we cannot solve the equation for x in terms of elementary functions of t. To find dx/dt, we realize that x *is* a function of t and just differentiate each term of the equation with respect to t (this is called implicit differentiation). We get

$$\frac{dx}{dt} + e^x\frac{dx}{dt} = 1. \tag{6.1}$$

Solving for dx/dt, we get

$$\frac{dx}{dt} = \frac{1}{1+e^x}.$$

Alternatively, we could use differentials here, and write first $dx + e^x dx = dt$; dividing by dt then gives (6.1).

We can also find higher derivatives by implicit differentiation (but do *not* use differentials for this since we have not given any meaning to the derivative or differential of a differential). Let us differentiate each term of (6.1) with respect to t; we get

(6.2)
$$\frac{d^2x}{dt^2} + e^x\frac{d^2x}{dt^2} + e^x\left(\frac{dx}{dt}\right)^2 = 0.$$

Solving for d^2x/dt^2 and substituting the value already found for dx/dt, we get

(6.3)
$$\frac{d^2x}{dt^2} = \frac{-e^x\left(\frac{dx}{dt}\right)^2}{1+e^x} = \frac{-e^x}{(1+e^x)^3}.$$

This problem is even easier if we want only the numerical values of the derivatives at a point. For $x = 0$ and $t = 1$, (6.1) gives

$$\frac{dx}{dt} + 1\cdot\frac{dx}{dt} = 1 \qquad \text{or} \qquad \frac{dx}{dt} = \frac{1}{2},$$

and (6.2) gives

$$\frac{d^2x}{dt^2} + 1\cdot\frac{d^2x}{dt^2} + 1\cdot\left(\frac{1}{2}\right)^2 = 0 \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -\frac{1}{8}.$$

Implicit differentiation is the best method to use in finding slopes of curves with complicated equations.

Example 2. Find the equation of the tangent line to the curve $x^3 - 3y^3 + xy + 21 = 0$ at the point $(1, 2)$.

We differentiate the given equation implicitly with respect to x to get

$$3x^2 - 9y^2\frac{dy}{dx} + x\frac{dy}{dx} + y = 0.$$

Substitute $x = 1, y = 2$:

$$3 - 36\frac{dy}{dx} + \frac{dy}{dx} + 2 = 0, \qquad \frac{dy}{dx} = \frac{5}{35} = \frac{1}{7}.$$

Then the equation of the tangent line is

$$\frac{y-2}{x-1} = \frac{1}{7} \qquad \text{or} \qquad x - 7y + 13 = 0.$$

By computer plotting the curve and the tangent line on the same axes, you can check to be sure that the line appears tangent to the curve.

Problems

1. If $pv^a = C$ (where a and C are constants), find dv/dp and d^2v/dp^2.

2. If $ye^{xy} = \sin x$ find dy/dx and d^2y/dx^2 at $(0,0)$.

3. If $x^y = y^x$, find dy/dx at $(2,4)$.

4. If $xe^y = ye^x$, find dy/dx and d^2y/dx^2 for $y \neq 1$.

5. If $xy^3 - yx^3 = 6$ is the equation of a curve, find the slope and the equation of the tangent line at the point $(1,2)$. Computer plot the curve and the tangent line on the same axes.

6. In Problem 5 find d^2y/dx^2 at $(1,2)$.

7. If $y^3 - x^2y = 8$ is the equation of a curve, find the slope and the equation of the tangent line at the point $(3,-1)$. Computer plot the curve and the tangent line on the same axes.

8. In Problem 7 find d^2y/dx^2 at $(3,-1)$.

9. For the curve $x^{2/3}+y^{2/3} = 4$, find the equations of the tangent lines at $(2\sqrt{2}, -2\sqrt{2})$, at $(8,0)$, and at $(0,8)$. Computer plot the curve and the tangent lines on the same axes.

10. For the curve $xe^y + ye^x = 0$, find the equation of the tangent line at the origin. *Caution:* Substitute $x = y = 0$ as soon as you have differentiated. Computer plot the curve and the tangent line on the same axes.

11. In Problem 10, find y'' at the origin.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).