The Second Derivative Test

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In single-variable calculus, there is a simple test to determine whether a given critical point is a local maximum or a local minimum:

When $f''(x_0) = 0$, the second derivative test is inconclusive.

Second Derivative Test (Single Variable)

Let $f(x)$ be a twice differentiable function, and let x_0 be a critical point for f.

1. If $f''(x_0) > 0$, then x_0 is a local minimum for f.
2. If $f''(x_0) < 0$, then x_0 is a local maximum for f.

This test can be generalized to multivariable functions as follows.

Though we are only stating this test for the two-variable case, it works for any number of variables.

When $Hf(x_0, y_0)$ is neither positive definite, negative definite nor indefinite, the second derivative test is inconclusive.

Second Derivative Test

Let $f(x, y)$ be a twice differentiable function, and let (x_0, y_0) be a critical point for f.

1. If $Hf(x_0, y_0)$ is positive definite, then (x_0, y_0) is a local minimum for f.
2. If $Hf(x_0, y_0)$ is negative definite, then (x_0, y_0) is a local maximum for f.
3. If $Hf(x_0, y_0)$ is indefinite, then (x_0, y_0) is a saddle point for f.

The reason that this test works is that the eigenvalues of the Hessian $H = Hf(x_0, y_0)$ are related to the directional second derivatives of f at x_0, y_0. In particular, if $\mathbf{u}$ is an eigenvector for H with eigenvalue λ, then

$$D_{\mathbf{u}} f(x_0, y_0) = \mathbf{u}^T H \mathbf{u} = \mathbf{u}^T \lambda \mathbf{u} = \lambda \mathbf{u}^T \mathbf{u} = \lambda.$$

Here $\mathbf{u}^T\mathbf{u} = 1$ since $\mathbf{u}$ is a unit vector.

That is, the directional derivative of the Hessian in the direction of an eigenvector $\mathbf{u}$ is equal to the corresponding eigenvalue. Thus we expect the eigenvalues of the Hessian to be positive at a local minimum and negative at a local maximum. Moreover, if the Hessian has both positive and negative eigenvalues, the corresponding point must be a saddle point.

It is less obvious that a critical point *must* be a local minimum just because all of the eigenvalues of the Hessian are positive. This argument requires some additional linear algebra that we will not pursue here.

EXAMPLE 5

The function $f(x, y) = x^3 + 2(x - y)^2 - 3x$ has a critical point at $(1, 1)$. Classify this critical point as a local maximum, a local minimum, or a saddle point.

SOLUTION The Hessian of f is

$$Hf(x, y) = \begin{bmatrix} 6x + 4 & -4 \\ -4 & 4 \end{bmatrix}$$

and in particular

$$Hf(1, 1) = \begin{bmatrix} 10 & -4 \\ -4 & 4 \end{bmatrix}$$

The eigenvalues add to 14 (the trace) and multiply to 24 (the determinant), so they must be 2 and 12.

The eigenvalues of this matrix are 2 and 12, so $(1, 1)$ is a local minimum.

EXAMPLE 6

The function $f(x, y) = 6\cos x + 4x\sin y$ has a critical point at $(0, 0)$. Classify this critical point as a local maximum, a local minimum, or a saddle point.

SOLUTION The Hessian of f is

$$Hf(x, y) = \begin{bmatrix} -6\cos x & 4\cos y \\ 4\cos y & -4x\sin y \end{bmatrix}$$

and in particular

$$Hf(0, 0) = \begin{bmatrix} -6 & 4 \\ 4 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are -8 and 2, so $(0, 0)$ is a saddle point.

The eigenvalues add to -6 (the trace) and multiply to -16 (the determinant), so they must be -8 and 2.

EXERCISES

1–2 ■ Compute the Hessian matrix for the given function f.

1. $f(x, y) = x^2 \sin y$

2. $f(x, y, z) = x^2 y^3 z^4$

3–4 ■ Compute the Hessian matrix for the given function f at the given point P.

3. $f(x, y) = x^3 + 4xy^2;\ P = (2, 3)$

4. $f(x, y, z) = \dfrac{16z}{\sqrt{xy}};\ P = (4, 1, 8)$

5. Let $f(x, y)$ be a twice differentiable function, and suppose that

$$Hf(x, y) = \begin{bmatrix} -2xy\sin(x^2) & \cos(x^2) \\ \cos(x^2) & 0 \end{bmatrix}.$$

Compute $f_{xy}(\sqrt{\pi}, 5)$.

6. Let $f(x, y) = x^3 + x^2 y$, and let $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(a) Find a formula for $D_{\mathbf{u}} f(x, y)$.

(b) Use your formula from part (a) to find a formula for $D_{\mathbf{u}}^2 f(x, y)$.

7. Let $f(x, y)$ be a twice differentiable function, and suppose that

$$Hf(2, 3) = \begin{bmatrix} 7 & 4 \\ 4 & 5 \end{bmatrix}.$$

Compute $D_{\mathbf{u}}^2 f(2, 3)$, where $\mathbf{u}$ is the unit vector $\mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$.

8. Let $f(x, y)$ be a twice differentiable function, and suppose that

$$Hf(x, y) = \begin{bmatrix} 0 & \sin(e^y) \\ \sin(e^y) & xe^y\cos(e^y) \end{bmatrix}.$$

Find a formula for $D_{\mathbf{u}}^2 f(x, y)$, where $\mathbf{u}$ is the unit vector $\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

9–12 ■ Find all critical points of the given function. (See Section 11.7 of the textbook.)

9. $f(x, y) = x^4 + y^4 - 4xy + 2$

10. $f(x, y) = x^3 - 12xy + 8y^3$

11. $f(x, y) = e^x \cos y$

12. $f(x, y) = e^y(y^2 - x^2)$

13–18 ■ A function and one of its critical points are given. Use the second derivative test to determine whether the critical point is a local maximum, a local minimum, or a saddle point.

13. $f(x, y) = \sin x \cos y;\ P = (\pi/2, 0)$

14. $f(x, y) = \sin x \cos y;\ P = (\pi/2, \pi)$

15. $f(x, y) = \sin x \cos y;\ P = (\pi, \pi/2)$

16. $f(x, y) = 7x^2 + 4xy + 4y^2 - 48x;\ P = (4, -2)$

17. $f(x, y) = 3x^2 + 4\cos(x + y);\ P = (0, 0)$

18. $f(x, y, z) = 3x^2 + (1 + z^2)\cos y;\ P = (0, 0, 0)$

19. Let $f(x, y) = x^3 - 3x^2 - 2y^2$. Find the critical points of f, and classify each critical point as a local maximum, a local minimum, or a saddle point.

- **References**

1. Boas, Mary L. *Mathematical methods in the physical sciences*. John Wiley & Sons, 2006.
2. Arfken George, Hans J. Weber, and F. Harris. "Mathematical Methods for Physicists. A Comprehensive Guide." (2013).