Lecture Eight

Unsteady State Heat Conduction

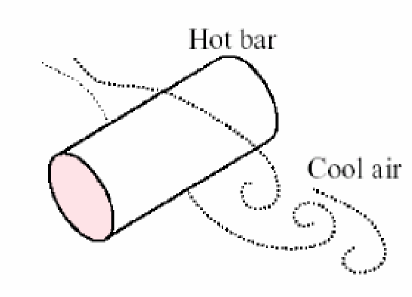
1- Unsteady Heat Transfer.

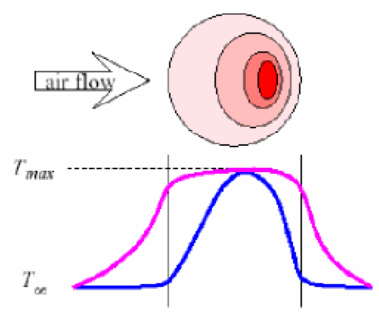
Many heat transfer problems require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T, \text{ or } \frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T$$

where $\alpha = \dfrac{k}{\rho c}$ is the thermal diffusivity

"A heated/cooled body at T_i is suddenly exposed to fluid at T_∞ with a known heat transfer coefficient . Either evaluate the temperature at a given time, or find time for a given temperature."





Q: "How good an approximation would it be to say the bar is more or less *isothermal*?"
A: "Depends on the relative importance of the thermal conductivity in the thermal circuit compared to the convective heat transfer coefficient".

2- **Biot Number (Bi).**

•Defined to describe the relative resistance in a thermal circuit of the convection compared

$$Bi = \frac{hL_c}{k} = \frac{L_c / kA}{1/hA} = \frac{\text{Internal conduction resistance within solid}}{\text{External convection resistance at body surface}}$$

L_c is a characteristic length of the body

Bi→0: No conduction resistance at all. The body is isothermal.

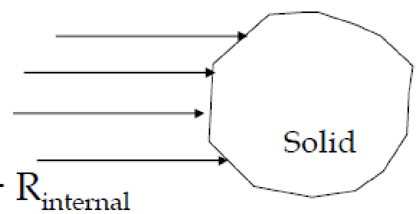
Small Bi: Conduction resistance is less important. The body may still be approximated as isothermal (purple temp. plot in figure) *Lumped capacitance analysis* can be performed.

Large Bi: Conduction resistance is significant. The body cannot be treated as isothermal (blue temp. plot in figure).

3- **Lumped Parameter Analysis.**

Transient heat transfer with no internal resistance.

Valid for Bi<0.1



Total Resistance= $R_{external} + R_{internal}$

GE: $$\frac{dT}{dt} = -\frac{hA}{mc_p}(T - T_\infty)$$ BC: $$T(t=0) = T_i$$

Solution: $let\ \Theta = T - T_\infty$, therefore

$$\frac{d\Theta}{dt} = -\frac{hA}{mc_p}\Theta$$

$$\Theta_i = T_i - T_\infty$$

$$\ln\frac{\Theta}{\Theta_i} = -\frac{hA}{mc_p}t$$

$$\frac{\Theta}{\Theta_i} = e^{-\frac{hA}{mc_p}t}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t \Big/ \frac{mc_p}{hA}}$$

- To determine the temperature at a given time, or
- To determine the time required for the temperature to reach a specified value.

> Note: Temperature function **only of time** and **not** of space!

$$\frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{hA}{\rho c V}t\right)$$

$$\frac{hA}{\rho c V}t = \left(\frac{hL_c}{k}\right)\left(\frac{k}{\rho c}\right)\frac{1}{L_c}\frac{1}{L_c}t = Bi\frac{\alpha}{L_c^{\,2}}t$$

Thermal diffusivity: $\alpha \equiv \left(\frac{k}{\rho c}\right)$ ($m^2\ s^{-1}$)

Define Fo as the Fourier number (dimensionless time)

$$Fo \equiv \frac{\alpha}{L_c^{\ 2}} t \quad \text{and Biot number} \quad Bi \equiv \frac{hL_C}{k}$$

The temperature variation can be expressed as

$$\mathbf{T} = \exp(-Bi*Fo)$$

where L_c is a characteristic length scale : realte to the size of the solid invloved in the problem

for example, $L_c = \frac{r_o}{2}$ (half - radius) when the solid is a cylinder.

$L_c = \frac{r_o}{3}$ (one - third radius) when the solid is sphere

$L_c = L$ (half thickness) when the solid is aplane wall with a 2L thickness

4- Analytical Solution.

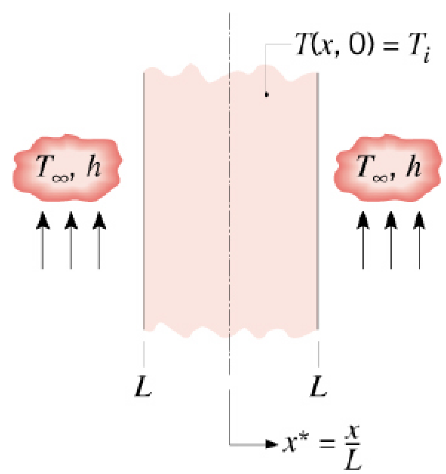
The Plane Wall: Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=l} = h\left[T(L,t) - T_\infty\right]$$



Note: Once spatial variability of temperature is included, there is existence of seven different independent variables.

How may the functional dependence be simplified?

- The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

Dimensionless temperature difference: $$\theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

Dimensionless coordinate: $x^* = \frac{x}{L}$

Dimensionless time: $t^* = \frac{\alpha t}{L^2} = Fo$

The Biot Number: $Bi = \frac{hL}{k_{solid}}$

The solution for temperature will now be a function of the other non-dimensional quantities

$$\theta^* = f(x^*, Fo, Bi)$$

Exact Solution: $$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad \zeta_n \tan\zeta_n = Bi$$

The roots (eigenvalues) of the equation can be obtained from tables given in standard textbooks.

The One-Term Approximation $Fo > 0.2$

Variation of mid-plane temperature with time Fo $\quad (x^* = 0)$

$$\theta_0^* = \frac{T - T_\infty}{T_i - T_\infty} \approx C_1 \exp\left(-\zeta_1^2 Fo\right)$$

From tables given in standard textbooks, one can obtain C_1 and ζ_1 as a function of Bi.

Variation of temperature with location (x^*) and time (Fo):

$$\theta^* = \theta_0^* = \cos\left(\zeta_1 x^*\right)$$

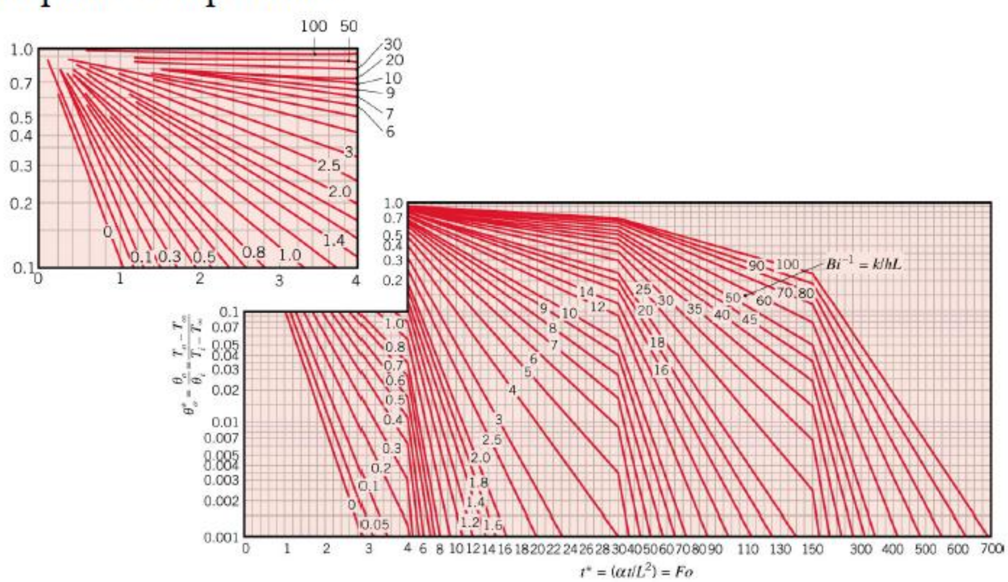
Change in thermal energy storage with time: $\Delta E_{st} = -Q$

$$Q = Q_0\left(1 - \frac{\sin\zeta_1}{\zeta_1}\right)\theta_0^*$$

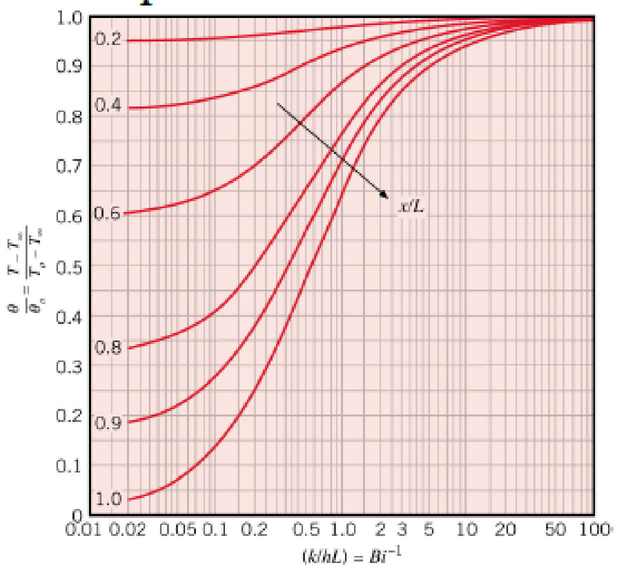
$$Q_0 = \rho c V\left(T_i - T_\infty\right)$$

5- <u>Graphical Representation (The Heisler Charts).</u>

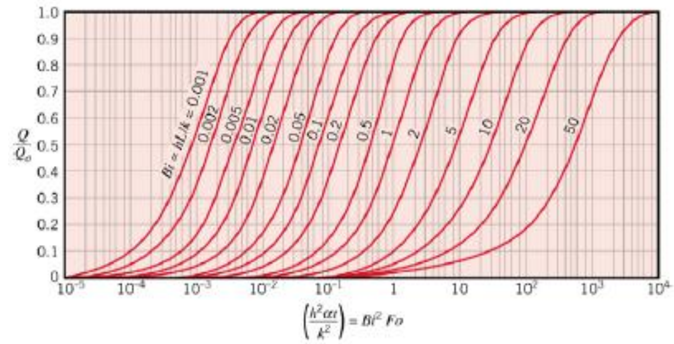
Midplane Temperature:



Temperature Distribution



Change in Thermal Energy Storage



Assumptions in using Heisler charts:

- Constant Ti and thermal properties over the body
- Constant boundary fluid T∞ by step change
- Simple geometry: slab, cylinder or sphere