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Lecture Seven

Numerical Technique solutions

1- Numerical Solution.

❑Matrix form: [**A**][**T**]=[**C**].

From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$

where $[\mathbf{A}]^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

❑ Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

❑ Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

❑ For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij}T_j + a_{ii}T_i + \sum_{j=i+1}^{N} a_{ij}T_j = C_i,$$

Replace (k) by (k-1) for the Jacobi iteration

$$(\text{Example}: a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i = 3)$$

Rewrite the equation of the form:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when max$|T_i(k)-T_i(k-1)| \le \varepsilon$, where ε specifies a predetermined value of acceptable error

Example

Solve the following system of equations using (a) the Jacobi methods, (b) the Gauss Seidel iteration method.

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix}$$

$$4X + 2Y + Z = 11,$$
$$-X + 2Y + 0 * Z = 3,$$
$$2X + Y + 4Z = 16$$

Reorganize into new form:

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z$$

$$Y = \frac{3}{2} + \frac{1}{2}X + 0 * Z$$

$$Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

- (a) Jacobi method: use initial guess X0=Y0=Z0=1,
- stop when max$|X_k-X_{k-1}, Y_k-Y_{k-1}, Z_k-Z_{k-1}| \le 0.1$
- First iteration:
- $X_1 = (11/4) - (1/2)Y_0 - (1/4)Z_0 = 2$
- $Y_1 = (3/2) + (1/2)X_0 = 2$
- $Z_1 = 4 - (1/2)X_0 - (1/4)Y_0 = 13/4$

Second iteration: use the iterated values $X^1=2$, $Y^1=2$, $Z^1=13/4$

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2) X^1 - (1/4)Y^1 = 5/2$

Converging Process:

$$[1,1,1],\ \left[2,2,\frac{13}{4}\right],\ \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right],\ \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right],\ \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right]$$. Stop the iteration when

$$\max\left|X^5 - X^4, Y^5 - Y^4, Z^5 - Z^4\right| \le 0.1$$

Final solution [1.014, 2.02, 2.996]

Exact solution [1, 2, 3]

(b) Gauss-Seidel iteration: Substitute the iterated values into the iterative process immediately after they are computed.

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z,\ Y = \frac{3}{2} + \frac{1}{2}X,\ Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$ Immediate substitution

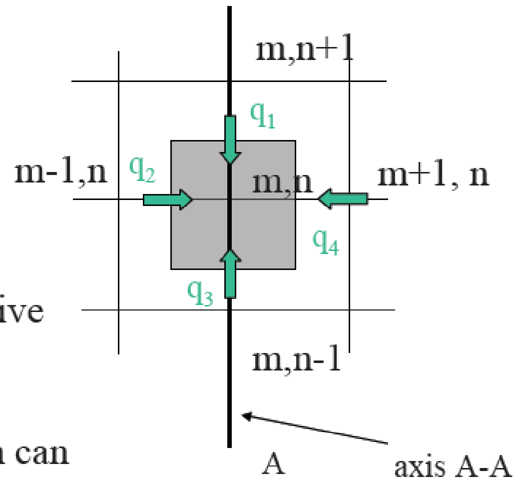
$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: $[1,1,1],\ \left[2,\frac{5}{2},\frac{19}{8}\right],\ \left[\frac{29}{32},\frac{125}{64},\frac{783}{256}\right],\ \left[\frac{1033}{1024},\frac{4095}{2048},\frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster

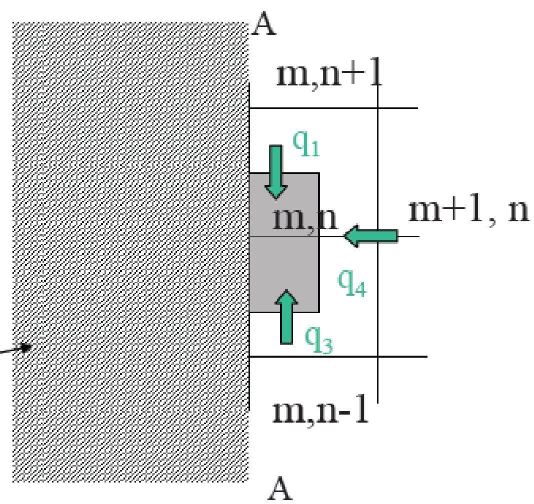
2- Numerical Method (Special Cases)



❑ For all the special cases discussed in the following, the derivation will be based on the standard nodal point configuration as shown to the right.

☐ Symmetric case: symmetrical relative to the A-A axis.
In this case, $T_{m-1,n}=T_{m+1,n}$
Therefore the standard nodal equation can be written as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}$$
$$=2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$



☐ Insulated surface case: If the axis A-A is an insulated wall, therefore there is no heat transfer across A-A. Also, the surface area for q_1 and q_3 is only half of their original value. Write the energy balance equation (q_2=0):

$$q_1+q_3+q_4=0$$

$$k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}+k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+k\Delta y\frac{T_{m+1,n}-T_{m,n}}{\Delta x}=0$$

$$2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation is identical to the symmetrical case discussed previously.

☐ With internal generation G=gV where g is the power generated per unit volume (W/m^3). Based on the energy balance concept:

$$q_1 + q_2 + q_3 + q_4 + G$$

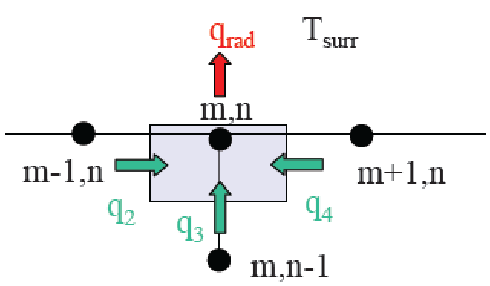
$$q_1 + q_2 + q_3 + q_4 + g(\Delta x)(\Delta y)(1) = 0$$

Use 1 to represent the dimension along the z-direction.

$$k(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n}) + g\left(\Delta x\right)^2 = 0$$

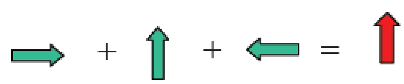
$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{g(\Delta x)^2}{k} = 0$$

☐ Radiation heat exchange with respect to the surrounding (assume no convection, no generation to simplify the derivation). Given surface emissivity ε, surrounding temperature T_{surr}.



From energy balance concept:

$q_2 + q_3 + q_4 = q_{rad}$



$$k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\Delta x\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k\left(\frac{\Delta y}{2}\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} = \varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}{}^4 - T_{surr}{}^4\right)$$

$$k\left(T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n}\right) = 2\varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}^4 - T_{surr}^4\right)$$

$$T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n} - \boxed{\frac{2\varepsilon\sigma\left(\Delta x\right)}{k}T_{m,n}^4} = -2\frac{\varepsilon\sigma\left(\Delta x\right)}{k}T_{surr}^4$$

Nonlinear term, cannot solve by using iteration technique.