1.1 First Order Ordinary Differential Equations

1.1.1 Variable Separable Differential Equations:

Any D.E that can be written in the form

$$\boxed{P(x)+Q(y).y'=0}$$

Is a separable equation, (because the dependent and independent variables are separated). We can obtain an implicit by integrating with respect to *x*.

$$\int P(x).dx+\int Q(y).\frac{dy}{dx}.dx=c$$

$$\boxed{\int P(x).dx+\int Q(y)dy=c}$$

Example: Consider the D.E $y'=xy^2$. We separate the dependent and independent variables and integrate to find the solution.

$$\frac{dy}{dx}=x.y^2$$
$$y^{-2}dy=x.dx$$
$$\int y^{-2}dy=\int x.dx+c$$
$$-y^{-1}=\frac{x^2}{2}+c$$
$$\left[y=\frac{-1}{x^2/2+c}\right]$$

Example: The equation $y'=y-y^2$ is separable.

$$\left(\frac{y'}{y-y^2}=1\right)$$

We expand in partial fraction and integrate.

$$\left(\frac{1}{y}-\frac{1}{y-1}\right).y'=1$$
$$\ln|y|-\ln|y-1|=x+c$$

We have an implicit function for *y(x)*. Now we solve for *y(x)*.

$$\ln\left|\frac{y}{y-1}\right| = x + c$$
$$\left|\frac{y}{y-1}\right| = e^{x+c}$$
$$\frac{y}{y-1} = \pm e^{x+c}$$
$$\frac{y}{y-1} = Ce^{x}$$

Example: Consider the D.E $(xy^2 + x)dx + (yx^2 + y)dy = 0$.We separate the dependant and independent variables and integrate to find the solution.

$$x(y^2+1).dx + y.(x^2+1)dy = 0$$
$$\frac{x.dx}{x^2+1} = -\frac{y.dy}{y^2+1}$$ Multiply by 2 and Integrate

$$\ln(x^2+1) + \ln(y^2+1) = c$$
$$\ln(x^2+1).(y^2+1) = c \Rightarrow (x^2+1).(y^2+1) = e^{c} = C$$

Example: Solve the following D.E ?

$$(4y - \cos y).\frac{dy}{dx} - 3x^2 = 0$$
$$(4y - \cos y).\frac{dy}{dx} = 3x^2$$
$$(4y - \cos y).dy = 3x^2.dx$$
$$\int (4y - \cos y).dy = \int 3x^2.dx$$
$$\frac{4y^2}{2} - \sin y = \frac{3x^3}{3} + c$$
$$2y^2 - \sin y = x^3 + c$$

(1.1.2) Exact differential equations:

Any first order ordinary D.E's of the first degree can be written as the total D.E,

$$P(x, y).dx + Q(x, y).dy = 0.$$

If this equation can be integrated directly, that is if there is a primitive, *u(x,y)*, such that,

$$du = P.dx + Q.dy,$$

then this equation is called *exact*. The (implicit) solution of the D.E is

$$u(x, y) = c,$$

where *c* is an arbitrary constant. Since the differential of a function, *u(x,y)*, is

$$du \equiv \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy,$$

P and *Q* are the partial derivatives of *u*:

$$P(x, y) = \frac{\partial u}{\partial x} \quad , \qquad Q(x, y) = \frac{\partial u}{\partial y} \ .$$

In an alternative notation, the D.E

$$\frac{du}{dx} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial x} = P(x, y) + Q(x, y).\frac{dy}{dx}$$

The solution of the D.E is *u(x,y) = c*.

Example:

$$x + y.\frac{dy}{dx} = 0$$

is an exact D.E since

$$\frac{d}{dx}\left(\frac{1}{2}\left(x^2 + y^2\right)\right) = x + y.\frac{dy}{dx}$$

The solution of the D.E is

$\frac{1}{2}(x^2 + y^2) = c$

Example: Let *f(x)* and *g(x)* be known functions.

$g(x).y' + g'(x).y = f(x)$

is an exact D.E since

$\frac{d}{dx}(g(x).y(x)) = g\,y' + g'\,y$.

The solution of D.E is

$g(x).y(x) = \int f(x).dx + c$

$y(x) = \frac{1}{g(x)}.\int f(x).dx + \frac{c}{g(x)}.$

A necessary condition for exactness. The solution of the Exact equation $P+Q.y'=0$ is $u=c$ where u is the primitive of the equation $\frac{du}{dx} = P + Q.y'$. At present the only method we have for determining the primitive is guessing. This is fine for simple equations, but for more difficult cases we would like a method more concrete than inspiration. As a first step toward this goal we determine a criterion for determining if an equation is exact.

Consider the exact equation,

$$\boldsymbol{P + Q \, . \, y' = 0} \, ,$$

with primitive *u*, where we assume that the function *P* and *Q* are continuously differentiable. Since the mixed partial derivatives of *u* are equal,

$\frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial^2 u}{\partial y . \partial x}$,

a necessary condition for exactness is $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.}$

Example: Prove that the following D.E is exact?

$y^2 dx + 2xy dy = 0$

$P = y^2$ $\qquad Q = 2xy$

$\frac{\partial P}{\partial y} = 2y$ $\qquad \frac{\partial Q}{\partial x} = 2y$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

Example: Prove that the following D.E is exact and find the general solution?

$$(3x^2 y + 2xy).dx + (x^3 + x^2 + 2y).dy = 0$$

$$P = 3x^2 y + 2xy \qquad Q = x^3 + x^2 + 2y$$

$$\frac{\partial P}{\partial y} = 3x^2 + 2x \qquad \frac{\partial Q}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$$

$$\frac{\partial u}{\partial x} = P = 3x^2 y + 2xy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial u}{\partial y} = Q = x^3 + x^2 + 2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

by integrating eq. (1) with respect to x we get

$$u = x^3 y + x^2 y + c$$

$$u = x^3 y + x^2 y + \phi(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

by deriving eq. (3) with respect to y we get

$$\frac{\partial u}{\partial y} = x^3 + x^2 + \phi'(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

and by equalizing eq. (4) with eq. (2) we get

$$x^3 + x^2 + \phi'(y) = x^3 + x^2 + 2y$$

$$\phi'(y) = 2y$$

$$\int \phi'(y) = \int 2y$$

$$\phi(y) = y^2 + c$$

and by substituting $\phi(y)$ *in eq. (3) we get the General solution as,*

$$u(x, y) = x^3 y + x^2 y + y^2 + c$$

Example: Prove that the following D.E is exact and find the general solution?

$(\cos x + y\sin x).dx = \cos x.dy$

$(\cos x + y\sin x).dx - \cos x.dy = 0$

$P = \cos x + y\sin x \qquad Q = -\cos x$

$\frac{\partial P}{\partial y} = \sin x \qquad \frac{\partial Q}{\partial x} = -(-\sin x) = \sin x$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

$\frac{\partial u}{\partial x} = P = \cos x + y\sin x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

$\frac{\partial u}{\partial y} = Q = -\cos x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$

$by \quad \text{int}egrating \quad eq. \quad (1)$

$u = \sin x - y\cos x + \phi(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$

$by \quad deriving \quad eq. \quad (3) \quad partially \quad to \quad y$

$\frac{\partial u}{\partial y} = -\cos x + \phi'(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$

$by \quad equalizing \quad eq. \quad (4) \quad to \quad eq. \quad (2) \quad we \quad get$

$-\cos x + \phi'(y) = -\cos x \quad \Rightarrow \quad \phi'(y) = 0 \quad \Rightarrow \quad \phi(y) = c$

$by \quad substituting \quad \phi(y) \quad in \quad eq. \quad (3)$

$u = \sin x - y\cos x + c \quad general \quad solution$

Example: Prove that the following D.E is exact and find the general solution?

$(xy\cos xy + \sin xy).dx + (x^2\cos xy + e^y).dy = 0$

$P = xy\cos xy + \sin xy \qquad Q = x^2\cos xy + e^y$

$\frac{\partial P}{\partial y} = -x.y.x.\sin xy + x.\cos xy + x.\cos xy \qquad \frac{\partial Q}{\partial y} = -x^2.y.\sin xy + 2x.\cos xy$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

$P = \frac{\partial u}{\partial x} = xy\cos xy + \sin xy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

$Q = \frac{\partial u}{\partial y} = x^2\cos xy + e^y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$

$$by \quad integrating \quad eq. \ (2) \quad with \quad respect \quad to \quad y$$

$$u = x.\sin xy + e^{y} + \phi(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$drive \quad eq. \ (3) \quad paritially \quad with \quad respect \quad to \quad x$$

$$\frac{\partial u}{\partial x} = x.y.\cos xy + \sin xy + \phi'(x) \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$by \quad equalizing \quad eq. \ (4) \quad with \quad eq. \ (1) \quad we \quad get$$

$$x.y.\cos xy + \sin xy + \phi'(x) = xy\cos xy + \sin xy \quad \Rightarrow \quad \phi'(x) = 0 \quad \Rightarrow \quad \phi(x) = c$$

$$\therefore \quad u = x.\sin xy + e^{y} + c \quad which \quad is \quad the \quad general \quad solution.$$

(1.1.3) Homogeneous differential equations:

Homogeneous coefficient, first order D.E's form another class of soluble eqs. We will find that a change in dependant variable will make such eqs. separable or we can determine an integrating factor that will make such eqs. exact. First we define homogeneous functions.

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

Example: Solve the homogeneous D.E?

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x} \quad \Rightarrow \quad y = v.x$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx} = f(v)$$

$$x.\frac{dv}{dx} = f(v) - v$$

$$\frac{x}{dx} = \frac{f(v) - v}{dv}$$

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\int \frac{dx}{x} + c = \int \frac{dv}{F(v) - v}$$