by integrating eq. (2) with respect to y

$$u = x.\sin xy + e^{y} + \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

drive eq. (3) paritially with respect to x

$$\frac{\partial u}{\partial x} = x.y.\cos xy + \sin xy + \phi'(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

by equalizing eq. (4) with eq. (1) we get

$$x.y.\cos xy + \sin xy + \phi'(x) = x y \cos xy + \sin xy \quad \Rightarrow \quad \phi'(x) = 0 \quad \Rightarrow \quad \phi(x) = c$$

$\therefore \quad u = x.\sin xy + e^{y} + c$ *which is the general solution.*

(1.1.3) Homogeneous differential equations:

Homogeneous coefficient, first order D.E's form another class of soluble eqs. We will find that a change in dependant variable will make such eqs. separable or we can determine an integrating factor that will make such eqs. exact. First we define homogeneous functions.

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

Example: Solve the homogeneous D.E?

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x} \quad \Rightarrow \quad y = v.x$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx} = f(v)$$

$$x.\frac{dv}{dx} = f(v) - v$$

$$\frac{x}{dx} = \frac{f(v) - v}{dv}$$

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\int \frac{dx}{x} + c = \int \frac{dv}{F(v) - v}$$

Example: Solve the homogeneous D.E?

$$y' = \frac{x^2 + y^2}{x.y}$$

$$\frac{dy}{dx} = \frac{x^2}{x.y} + \frac{y^2}{x.y}$$

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx}$$

$$substitute \quad in \quad Eq. \quad (1)$$

$$v + x.\frac{dv}{dx} = \frac{1}{v} + v$$

$$x\frac{dv}{dx} = \frac{1}{v} \quad \Rightarrow \quad \frac{x}{dx} = \frac{1}{v.dv}$$

$$\int \frac{dx}{x} = \int v.dv$$

$$\ln x = \frac{v^2}{2} \quad \Rightarrow 2(\ln x + c_1) = v^2$$

$$2(\ln x + c_1) = \left(\frac{y}{x}\right)^2 \quad \Rightarrow 2\ln x + 2c_1 = \frac{y^2}{x^2}$$

$$\ln x^2 + C = \frac{y^2}{x^2} \quad \Rightarrow y^2 = x^2(\ln x^2 + C)$$

Example: Solve the homogeneous D.E?

$$2xyy' - y^2 + x^2 = 0$$

$$\frac{2xy}{x^2}.\frac{dy}{dx} - \frac{y^2}{x^2} + \frac{x^2}{x^2} = 0$$

$$\frac{2y}{x}.\frac{dy}{dx} - \left(\frac{y}{x}\right)^2 + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + x.\frac{dv}{dx} \Rightarrow 2v\left(v + x.\frac{dv}{dx}\right) - v^2 + 1 = 0$$

$$2.v^2 + 2.x.v.\frac{dv}{dx} - v^2 + 1 = 0 \Rightarrow 2.x.v.\frac{dv}{dx} + v^2 + 1 = 0$$

$$2.x.v.\frac{dv}{dx} = -(v^2+1) \Rightarrow \frac{x}{dx} = \frac{-(v^2+1)}{2.v.dv}$$

$$\frac{dx}{x} = \frac{2.v.dv}{-(v^2+1)} \Rightarrow -\int\frac{dx}{x} = \int\frac{2.v.dv}{v^2+1}$$

$$-\ln x + c = \ln(v^2+1) \Rightarrow \ln x^{-1} + c = \ln(v^2+1)$$

$$e^{(\ln x^{-1}+c)} = e^{\ln(v^2+1)} \Rightarrow e^{\ln x^{-1}}.e^c = e^{\ln(v^2+1)}$$

$$x^{-1}.c = v^2 + 1 \Rightarrow \frac{c}{x} = \left(\frac{y}{x}\right)^2 + 1$$

$$\frac{c}{x} = \frac{y^2}{x^2} + 1$$

(1.1.3.1) Equations reducible to homogeneous form:

Certain equations of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$$

can be reduced to the homogeneous form by substitution of

$$x = X + h \qquad y = Y + k, \quad then \quad \frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C} = \frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C} \quad where,$$

$$ah + bk + c = 0$$
$$Ah + Bk + C = 0$$

Example: Solve the following D.E?

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

$assume \quad x = X + h \qquad y = Y + k$

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3} = \frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$$

$$h + 2k - 3 = 0$$

$$\underline{2h + k - 3 = 0}$$

$$3k - 6 + 3 - 3 = 0 \quad \Rightarrow \quad k = 1 \quad \Rightarrow \quad h = 1$$

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \quad , \quad Y = VX$$

$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$$

$$X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1-V^2}{2+V}$$

$$\frac{(2+V)\,dV}{1-V^2} = \frac{dX}{X} \quad \Rightarrow \quad \frac{2+V}{1-V^2} = \frac{A}{1-V} + \frac{B}{1+V} = \frac{A(1+V)+B(1-V)}{1-V^2}$$

$$2 + V = A(1+V) + B(1-V)$$

$$2 + V = V(A-B) + A + B$$

$$A - B = 1$$

$$\underline{A + B = 2}$$

$$2A = 3 \quad \Rightarrow \quad A = \frac{3}{2} \quad , \quad B = \frac{1}{2}$$

$$\frac{\frac{3}{2}dV}{1-V} + \frac{\frac{1}{2}dV}{1+V} = \frac{dX}{X} \quad \Rightarrow \quad \frac{3\,dV}{2(1-V)} + \frac{dV}{2(1+V)} = \frac{dX}{X}$$

$$\frac{3\,dV}{(1-V)} + \frac{dV}{(1+V)} = \frac{2\,dX}{X}$$

$$-3\ln(1-V) + \ln(1+V) = 2\ln X$$

$$\ln(1-V)^{-3} + \ln(1+V) - \ln X^2 = C$$

$$\frac{\ln(1+V)}{\ln(1-V)^3} - \ln X^2 = C \quad \Rightarrow \quad \ln\frac{1+V}{X^2(1-V)^3} = C$$

$$\frac{1+V}{X^2(1-V)^3} = c \quad \Rightarrow \quad \frac{1+\frac{Y}{X}}{X^2(1-\frac{Y}{X})^3} = c \quad \Rightarrow \quad \frac{\frac{X+Y}{X}}{X^2\,\frac{(X-Y)^3}{X^3}} = c \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = c$$

$$x = X + h = X + 1 \quad \Rightarrow \quad X = x - 1 \quad \Rightarrow \quad Y = y - 1$$

$$\therefore \quad \frac{x-1+y+1}{[x-1-(y-1)]^3} = c \quad \Rightarrow \quad \frac{x+y-2}{(x-y)^3} = c$$