(1.1.4) The First Order, Linear Differential Equations:

(1.1.4.1) Homogeneous Equations:

The first order, linear, homogeneous D.E has the form

$$\boxed{\frac{dy}{dx} + p(x).y = 0}$$

We can solve any equation of this type because it is separable.

$$\frac{y'}{y} = -p(x)\,dx$$

$$\ln|y| = -\int p(x).dx + c$$

$$y = \pm e^{-\int p(x).dx + c}$$

$$\boxed{y = ce^{-\int p(x).dx}}$$

Example: Consider the equation

$$\frac{dy}{dx} + \frac{1}{x}y = 0$$

$$y = ce^{-\int p(x).dx} \quad \Rightarrow \quad y(x) = ce^{-\int \frac{1}{x}dx} \quad , \quad for \quad x \neq 0$$

$$y(x) = ce^{-\ln|x|}$$

$$y(x) = \frac{c}{|x|} \quad \Rightarrow \quad y(x) = \frac{c}{x}$$

(1.1.4.2) Inhomogeneous Equations:

The first order, linear, inhomogeneous D.E has the form

$$\frac{dy}{dx} + p(x).y = f(x)$$

There are two ways for linear inhomogeneous D.E.

1) $\frac{dy}{dx} + p(x).y = f(x)$

   the solution of this D.E is:

   $I(x) = e^{\int p(x)dx}$

   $I(x).y = \int I(x)f(x)\,dx + c$

2) $\frac{dx}{dy} + p(y).x = f(y)$

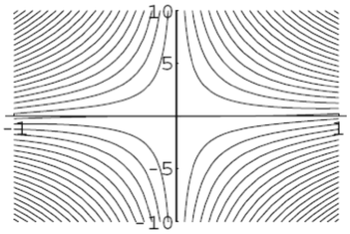
   the solution of this D.E is:

   $I(y) = e^{\int p(y)dy}$

   $I(y).x = \int I(y)f(y)\,dy + c$

Example1: consider the D.E

$$y' + \frac{1}{x}y = x^2 \quad , \quad x \succ o.$$



Solution to y' + y/x =x²

First find the integrating factor.

$$I(x) = \exp\left(\int \frac{1}{x}dx\right) = e^{\ln x} = x$$

then, multiply by the integrating factor and integrate.

$$I(x).y = \int I(x)f(x)\,dx + c$$

$$x.y = \int x.x^2\,dx + c = \int x^3 dx + c = \frac{1}{4}x^4 + c$$

$$y = \frac{1}{4}x^3 + \frac{c}{x}$$

Note that the general solution to the D.E is a one-parameter family of functions. The general solution is plotted in the figure above for various values of *c*.

Example2: Solve the following linear D.E?

$$\frac{dy}{dx}+2y=\cos x$$

$$p(x)=2 \qquad f(x)=\cos x$$

$$I(x)=e^{\int p(x).dx}=e^{\int 2.dx}=e^{2x}$$

$$I(x).y=\int I(x).f(x).dx+c$$

$$e^{2x}.y=\int e^{2x}.\cos x.dx+c \quad \text{..........................(1)}$$

$$\int e^{2x}.\cos x.dx=u.v-\int v.du \quad integration \quad by \quad part$$

$$Let \quad u=e^{2x} \quad \Rightarrow \quad du=2e^{2x}.dx$$

$$dv=\cos x.dx \quad \Rightarrow \quad v=\int \cos x.dx=\sin x$$

$$\therefore \quad \int e^{2x}.\cos x.dx=e^{2x}.\sin x-2\int e^{2x}.\sin x.dx$$

$$Also \quad \int e^{2x}.\sin x.dx=u.v-\int v.du \quad integration \quad by \quad part$$

$$Let \quad u=e^{2x} \quad \Rightarrow \quad du=2e^{2x}.dx$$

$$dv=\sin x.dx \quad \Rightarrow \quad v=\int \sin x.dx=-\cos x$$

$$\therefore \quad \int e^{2x}.\sin x.dx=-e^{2x}.\cos x-\int -\cos x.2e^{2x}.dx$$

$$\int e^{2x}.\sin x.dx=-e^{2x}.\cos x+2\int -\cos x.e^{2x}.dx$$

$$\int e^{2x}.\cos x.dx=e^{2x}.\sin x-2\left[-e^{2x}.\cos x+2\int e^{2x}.\cos x.dx\right]$$

$$\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x-4\int e^{2x}.\cos x.dx$$

$$\int e^{2x}.\cos x.dx+4\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x$$

$$5\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x$$

$$\therefore \quad \int e^{2x}.\cos x.dx=\frac{e^{2x}.\sin x+2e^{2x}.\cos x}{5}$$

$$substitute \quad in \quad eq. \quad (1)$$

$$e^{2x}.y=\frac{e^{2x}.\sin x+2e^{2x}.\cos x}{5}+c$$

Example3: Solve the following linear D.E?

$$\frac{dy}{dx}+2y=e^{-x}$$

Sol/

$$p(x)=2 \qquad f(x)=e^{-x}$$

$$I(x)=e^{\int p(x)dx}=e^{\int 2\,dx}=e^{2x}$$

$$I(x).y = \int I(x).f(x)dx + c$$

$$e^{2x}.y = \int e^{2x}.e^{-x}dx = \int e^{x}dx = e^{x} + c$$

$$e^{2x}.y = e^{x} + c$$

Example4: Solve the following linear D.E?

$$2\,\frac{dy}{dx} - y = e^{x/2}$$

Sol/

$$\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2}$$

$$p(x) = -\frac{1}{2} \qquad f(x) = \frac{1}{2}e^{x/2}$$

$$I(x) = e^{\int p(x)dx} = e^{\int -1/2\,dx} = e^{-x/2}$$

$$I(x).y = \int I(x).f(x)dx + c$$

$$e^{-x/2}.y = \int e^{-x/2}.\frac{1}{2}e^{x/2} = \frac{x}{2} + c$$

$$e^{-x/2}.y = \frac{x}{2} + c$$

Example5: Solve the following linear D.E?

$$x\,dy + y\,dx = \sin x\,dx$$

Sol/

$$x\,dy + y\,dx - \sin x\,dx = 0$$

$$x\,dy + (y - \sin x)\,dx \qquad \backslash\, x.dx$$

$$\frac{dy}{dx} + \frac{y - \sin x}{x} = 0$$

$$\frac{dy}{dx} + \frac{y}{x} - \frac{\sin x}{x} = 0$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x}$$

$$p(x) = -\frac{1}{x} \qquad f(x) = \frac{\sin x}{x}$$

$$I(x) = e^{\int p(x)dx} = e^{\int -1/x\,dx} = e^{\ln(x)} = x$$

$$I(x).y = \int I(x).f(x)dx + c$$

$$x.y = \int x\frac{\sin x}{x}dx = \int \sin x\;dx = -\cos x + c$$

$x.y = -\cos x + c$

Example6: Solve the following linear D.E?

$$(x - 2y)\, dy + y\, dx = 0$$

Sol/

$(x - 2y)\, dy + y\, dx = 0 \quad$ / *(x-2y) dx*

$$\frac{dy}{dx} + \frac{y}{x - 2y} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x - 2y}$$

$$\frac{dx}{dy} = -\frac{x - 2y}{y}$$

$$\frac{dx}{dy} = -\frac{x - 2y}{y}$$

$$\frac{dx}{dy} = -\frac{x}{y} + \frac{2y}{y} = -\frac{x}{y} + 2$$

$$\frac{dx}{dy} + \frac{1}{y}x = 2$$

$$p(y) = -\frac{1}{y} \qquad f(y) = 2$$

$$I(y) = e^{\int p(y)dy} = e^{\int 1/y\, dy} = e^{\ln(y)} = y$$

$$I(y).x = \int I(y).f(y)dy + c$$

$$y.x = \int y.2\, dy + c = y^2 + c$$

$$y.x = y^2 + c$$

(1.1.4.3) Equation reducible to liner form (Bernoulli's equation):

The Eq. of the form $\frac{dy}{dx} + P.y = Q.y^n$ can be reduced to linear form by dividing by y^n and substituting $z = y^{1-n}$.

$$\frac{dy}{dx} + P.y = Q.y^n$$

$$y^{-n}.y' + P.y^{1-n} = Q,$$

$$\because \quad z = y^{1-n} \quad \therefore \quad \frac{dz}{dx} = (1-n).y^{-n}$$

$$y^{-n}.y' = \frac{1}{1-n}.\frac{dz}{dx} \quad \Rightarrow \quad \frac{1}{1-n}.\frac{dz}{dx} + P.z = Q$$

Example1: solve the following D.E?

$$\frac{dy}{dx} - x.y = -y^3.e^{-x^2}$$

$$y^{-3}.y' - x.y^{-2} = -e^{-x^2}$$

$$z = y^{-2} \quad \Rightarrow \quad \frac{dz}{dx} = -2y^{-3}.y'$$

$$y^{-3}.y' = -\frac{1}{2}.\frac{dz}{dx} \quad \Rightarrow \quad -\frac{1}{2}.\frac{dz}{dx} - x.z = -e^{-x^2} \qquad *\text{-2}$$

$$\frac{dz}{dx} + 2x.z = 2e^{-x^2} \quad which \quad is \quad a\,linear \quad eq.$$

Solve and re-substitute in the first eq.

Example2: solve the following D.E?

$$2\frac{dy}{dx} - \frac{y}{x} = -y^3 \cos x \qquad \div y^3 \quad \textbf{Bernoulli equation}$$

$$2y^{-3}\frac{dy}{dx} - \frac{1}{x}y^{-2} = -\cos x$$

$$let \quad z = y^{1-n} = y^{1-3} = y^{-2}$$

$$\frac{dz}{dy} = -y^{-3}\frac{dy}{dx}$$

$$-\frac{dz}{dy} - \frac{1}{x}z = -\cos x \qquad * -1$$

$$\frac{dz}{dy} + \frac{1}{x}z = \cos x \quad \text{linear D.E}$$

$$p(x) = \frac{1}{x} \qquad f(x) = \cos x$$

$$I(x) = e^{\int p(x)dx} = e^{\int 1/x\, dx} = e^{\ln(x)} = x$$

$$I(x).z = \int I(x)\,f(x)dx$$

By part u=x du=dx
dv=cos(x) v=sin(x)

$$x.z = \int x.\cos x\, dx$$

$$x.z = x.\sin x - \int \sin x\, dx$$