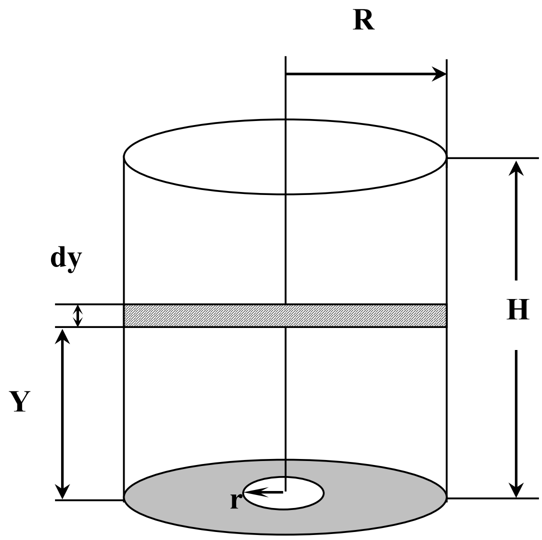
$x.z = x.\sin x + \cos x + c$
$x.y^{-2} = x.\sin x + \cos x + c$

Applications of Firs Order

Ordinary Differential Equations

Example: A cylindrical tank of radius R and height H initially filled with water. At the bottom of the tank there is a hole of radius r , through which water drains under the influence of gravity. Find the depth of water at any time t , and determine how long it takes the tank to drain off completely?

Solution: $dV = -\Pi . R^2 . dy$



$$v(velocity) = \sqrt{2gh} = \sqrt{2gy}$$

$$\frac{dV}{dt} = Q = A \times v \quad \Rightarrow \quad Q = \Pi r^2 \sqrt{2gy}$$

$$dV = \Pi r^2 \sqrt{2gy}.dt$$

$$\Pi r^2 \sqrt{2gy}.dt = -\Pi . R^2 .dy$$

$$r^2 \sqrt{2gy}.dt = -R^2 .dy$$

$$\frac{dy}{\sqrt{2gy}} = -\frac{r^2}{R^2}.dt \quad \Rightarrow \quad -\frac{dy}{\sqrt{2gy}} = \frac{r^2}{R^2}.dt$$

$$-(2gy)^{-\frac{1}{2}}.dy = \frac{r^2}{R^2}.dt$$

$$\frac{2g}{2g}\int -(2gy)^{-\frac{1}{2}}.dy = \int \frac{r^2}{R^2}.dt$$

$$-\frac{(2gy)^{\frac{1}{2}}}{\frac{1}{2}\times 2g} = \frac{r^2}{R^2}.t + c$$

$$-\frac{\sqrt{2gy}}{g} = \frac{r^2}{R^2}.t + c$$

$$+\sqrt{\frac{2y}{g}} = -\frac{r^2}{R^2}.t + c \quad initial\ conditions, \quad y = H \quad at \quad t = 0$$

$$\sqrt{\frac{2H}{g}} = 0 + c \quad \Rightarrow \quad c = \sqrt{\frac{2H}{g}}$$

$$\sqrt{\frac{2y}{g}} = -\frac{r^2}{R^2}.t + \sqrt{\frac{2H}{g}}$$

To find the required time for the tank to drain completely (t_o), we substitute ***y = 0***,

$$\sqrt{\frac{2y}{g}} = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

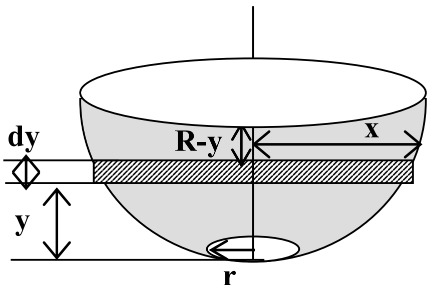
$$0 = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

$$t_o = \sqrt{\frac{2H}{g}}.\frac{R^2}{r^2}$$

Example: A spherical (half ball) tank of radius R initially filled with water. At the bottom of the tank there is a hole of radius r, through which water drains under the influence of gravity. Find the depth of water at any time t, and determine how long it takes the tank to drain off completely?

Solution: $dV = -\Pi x^2\,dy$ water loose

$dV = \sqrt{2gy}.\,\Pi r^2\,dt$ water outlet



$$R^2 = x^2 + (R-y)^2$$
$$x^2 = R^2 - (R-y)^2$$
$$x^2 = R^2 - (R^2 - 2Ry + y^2)$$
$$x^2 = 2Ry - y^2$$

$$\sqrt{2gy}.\,\Pi r^2\,dt = -\Pi x^2\,dy$$

$$\sqrt{2gy}.\,r^2\,dt = -x^2\,dy$$

$$\sqrt{2gy}.\;r^2\,dt = (y^2 - 2Ry)\,dy$$

$$\sqrt{2g}\,.\;r^2\,dt = \frac{(y^2 - 2Ry)\,dy}{\sqrt{y}}$$

$$\int \sqrt{2g}\,.\;r^2\,dt = \int (y^{3/2} - 2Ry^{1/2})\,dy$$

$$\sqrt{2g}\,.\;r^2.t = \frac{2}{3}y^{5/2} - 2Ry^{3/2} \times \frac{2}{3} + c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{2/3} = \sqrt{2g}\,.\;r^2.t + c$$

$$initial\ condition: at \quad t = 0 \quad y = R$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}RR^{2/3} = 0 + c$$

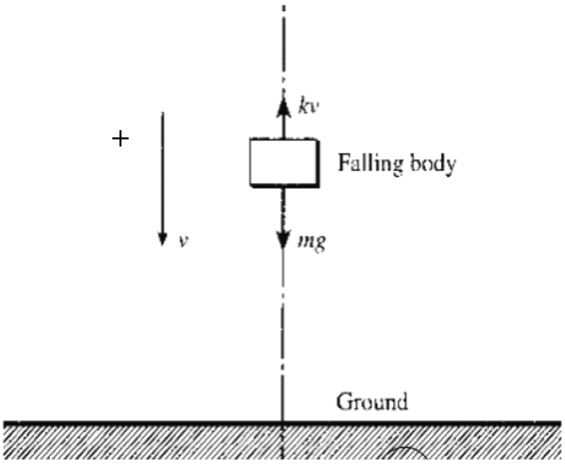
$$(\frac{2}{5} - \frac{4}{3})R^{5/2} = c$$

$$c = -\frac{14}{15}R^{5/2} = c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{3/2} = \sqrt{2g}\,.\;r^2.t + \frac{14}{15}R^{5/2}$$

Example: a body falls in a medium with resistance proportional to speed at any instant. If the limiting speed is 50 *ft/sec* and the speed of the body decreases to half (25 *ft/sec*) after (1 *sec*), what was the *initial velocity*?

Solution:



$$Force = mass \times acceleration$$

$$F = m\frac{dv}{dt}, \quad Newton's\ second\ law$$

$$m\,g - k\,v = m\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{k}{m}v = g \quad \Rightarrow \quad linear\ differential\ equation$$

$$Q = g \quad , \quad P = \frac{k}{m}$$

$$I = e^{\int p\,dt} \quad \Rightarrow \quad I = e^{\int \frac{k}{m}dt} \quad \Rightarrow \quad I = e^{\frac{k}{m}.t}$$

$$I.y = \int I.Q.dt \quad \Rightarrow \quad I.v = \int I.Q.dt$$

$$e^{\frac{k}{m}.t}.v = \int e^{\frac{k}{m}.t}.g.dt + c$$

$$e^{\frac{k}{m}.t}.v = g.\frac{m}{k}.e^{\frac{k}{m}.t} + c \qquad and\ dividing\ by\ e^{\frac{k}{m}.t}$$

$$v = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.t}}$$

$\underline{Initial\ conditions:}$

$$at\ t = \infty \quad \Rightarrow \quad v = 50\ ft/\sec$$

$$50 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.\infty}} \quad \Rightarrow \quad 50 = \frac{g.m}{k} + \frac{c}{\infty} \quad \Rightarrow \quad k = \frac{g.m}{50}$$

$$at\ t = 1 \quad \Rightarrow \quad v = 25\ ft/\sec$$

$$25 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.1}} \quad \Rightarrow \quad 25 = \frac{g.m}{\frac{g.m}{50}} + \frac{c}{e^{\frac{g.m}{50.m}}}$$

$$c = -25 \times e^{\frac{g}{50}}$$