Chapter 2

Second Order Ordinary Differential Equations

2. Linear Second Order Differential Equations with constant coefficients.

The General formula of a linear D.Es is:

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots\dots + P_n \; y = f(x)$$

where P_1 , P_2 *……* P_n *,* $f(x)$ *are functions of* (x)

The General formula of a linear D.E with constant coefficients is:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots\dots + a_n \; y = f(x)$$

If $f(x) = 0$ then the equation is **Homogeneous**.
If $f(x) \neq 0$ then the equation is **Nonhomogeneous**.

Examples:

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 5\,y = x \quad \Rightarrow \quad \text{Nonhomogene}$$

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 5\,y = 0 \quad \Rightarrow \quad Homogeneous$$

Using Operator:

$$\frac{d}{dx} = Operater = D \;\rightarrow \frac{dy}{dx} = Dy$$

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$(D^2 + 5D + 6)y = 0$$

if $y=sin(x)$

$$if\ y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2 y}{dx^2} = -\sin x$$

2.1 Solution of Homogeneous Linear D.Es with constant coefficients:

Example:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \quad \Longrightarrow (D^2 + a_1D + a_2)y = 0$$

General solution for second-order homogeneous linear D.Es:

Case 1: ***If m_1 and m_2 are real and distinct $m_1 \neq m_2$***

$$(D^2 + a_1 D + a_2)\, y = 0$$

$$(D - m_1)(D - m_2)\, y = 0$$

$$let \quad u = (D - m_2)\, y \quad \Rightarrow \quad (D - m_1)u = 0$$

$$D u = m_1 u \quad \Rightarrow \frac{du}{dx} = m_1 u \quad \Rightarrow \quad \int \frac{du}{u} = \int m_1\, dx + c$$

$$\ln u = m_1\, x + c \quad \Rightarrow \quad u = e^{m_1\, x + c} = c_1\, e^{m_1\, x}$$

$$u = (D - m_2)\, y \quad \Rightarrow \quad (D - m_2)\, y = c_1\, e^{m_1\, x}$$

$$\frac{dy}{dx} - m_2\, y = c_1\, e^{m_1\, x} \quad \Leftrightarrow \quad linear\ D.E$$

$$\frac{dy}{dx} + P\, y = Q \quad \Rightarrow \quad P = -m_2 \qquad Q = c_1\, e^{m_1\, x}$$

$$I(x) = \int e^{p(x)\, dx} = \int e^{-m_2\, dx} = e^{-m_2\, x}$$

$$I(x)\, y = \int I(x).Q.dx + c_2$$

$$e^{-m_2\, x}.y = \int e^{-m_2\, x}.c_1\, e^{m_1\, x}.dx + c_2$$

$$e^{-m_2\, x}.y = \int c_1\, e^{(m_1 - m_2).x}.dx + c_2$$

$$e^{-m_2\, x}.y = \frac{1}{m_1 - m_2}.\int c_1\, e^{(m_1 - m_2).x}.dx + c_2$$

$$e^{-m_2\, x}.y = \frac{c_1\, e^{(m_1 - m_2).x}}{m_1 - m_2} + c_2$$

$$y = \frac{c_1\, e^{(m_1 - m_2).x}}{m_1 - m_2}.e^{m_2\, x} + c_2.e^{m_2\, x} \quad \Rightarrow \quad y = \frac{c_1\, e^{m_1\, x - m_2\, x + m_2\, x}}{m_1 - m_2} + c_2.e^{m_2\, x}$$

$$y = C_1e^{m_1x} + C_2e^{m_2x} \quad where \quad m_1 \neq m_2$$

Case 2: ***If $m_1 = m_2$,***

$$(D-m)^2\, y = 0$$
$$(D-m)(D-m)\, y = 0$$
$$Let\ z = (D-m)\, y \quad \Rightarrow \quad (D-m)\, z = 0$$
$$\frac{dz}{dx} - m\, z = 0 \quad \Rightarrow \quad \frac{dz}{dx} = m\, z \quad \Rightarrow \quad \frac{dz}{z} = m\, dx \quad \Rightarrow \quad \ln z = m\, x + c_1$$
$$z = e^{m\, x}.e^{c_1} = C_1\, e^{m\, x}$$
$$\because \quad (D-m)\, y = z \quad \Rightarrow \quad (D-m)\, y = C_1\, e^{m\, x} \quad \Rightarrow \quad \frac{dy}{dx} - m\, y = C_1\, e^{m\, x} \quad \Leftrightarrow \quad linear$$
$$P = -m \qquad Q = C_1\, e^{m\, x}$$
$$I(x) = \int e^{p(x)\, dx} = \int e^{-m\, dx} = e^{-m\, x}$$
$$I(x)\, y = \int I(x).Q.dx + c_2$$
$$e^{-m\, x}.y = \int e^{-m\, x}.C_1\, e^{m\, x}.dx + c_2$$
$$e^{-m\, x}.y = (C_1\, x + c_2) \quad \Rightarrow$$

$$y = (C_1 x + C_2).e^{mx} \qquad where \quad m_1 = m_2$$

Case 3: ***If $m_1 = p + q\, i$ and $m_2 = p - q\, i$ the characteristic equation has a complex roots.***

$$Let\ the\ general\ \text{hom}ogeneous\ Linear\ D.E\ form\ be:$$
$$y'' + p_1\, y' + p_2\, y = 0$$
$$m_{1,2} = \frac{-p_1 \mp \sqrt{p_1^{\,2} - 4\, p_2}}{2}$$
$$Let\ p_1^{\,2} - 4\, p_2 = -4\, q^2$$
$$m_{1,2} = -\frac{p_1}{2} \mp i\, q$$
$$y = c_1\, e^{(-\frac{p_1}{2} - i\, q)\, x} + c_2\, e^{(-\frac{p_1}{2} + i\, q)\, x}$$
$$y = e^{-\frac{p_1}{2} x}\left[c_1\, e^{-i\, q\, x} + c_2\, e^{i\, q\, x}\right]$$
$$\sin x = \frac{e^{i\, x} - e^{-ix}}{2i} \quad and \quad \cos x = \frac{e^{i\, x} + e^{-ix}}{2i} \quad Euler's\ formula$$
$$e^{i\, x} = \cos x + i.\sin x$$
$$e^{-ix} = \cos x - i.\sin x$$

$$y = e^{-\frac{p_1}{2}x}\left[c_1(\cos qx - i.\sin qx) + c_2(\cos qx + i.\sin qx)\right]$$
$$y = e^{-\frac{p_1}{2}x}\left[(c_1 + c_2)\cos qx + (c_1 - c_2)\sin qx\right]$$
$$\therefore y = e^{-\frac{p_1}{2}x}(A\cos qx + B\sin qx)$$

$$y = e^{px}(A\cos qx + B\sin qx) \quad where \quad m_{1,2} = p + qi$$

Example1: Find the general solution of the equation: $y''+7y'+12y = 0$?

Solution: $y''+7y'+12y = 0 \Rightarrow m^2 + 7m + 12 = 0$

$$m_{1,2} = \frac{-7 \mp \sqrt{49-48}}{2} \Rightarrow m_1 = -4 \qquad m_2 = -3$$
$$y = C_1 e^{m_1 x} + c_2.e^{m_2 x} = C_1 e^{-4x} + c_2.e^{-3x}$$

Example2: Solve the equation: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$?

Solution: $m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \Rightarrow m_1 = 3 \qquad m_2 = 2$

$$y = C_1 e^{m_1 x} + c_2.e^{m_2 x} = C_1 e^{3x} + c_2.e^{2x}$$

Example3: Solve the equation: $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$?

Solution:

$$4m^2 + 4m + 1 = 0 \Rightarrow m^2 + m + \frac{1}{4} = 0 \Rightarrow (2m+1)^2 = 0 \Rightarrow m_1 = m_2 = -\frac{1}{2}$$
$$y = (C_1 x + c_2).e^{mx} \Rightarrow y = (c_1 x + c_2)\, e^{-\frac{1}{2}x}$$

Example4: Solve the equation: $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$?

Solution:

$$2m^2 + 3m + 4 = 0$$
$$m_{1,2} = \frac{-3 \mp \sqrt{9-(4\times2\times4)}}{2\times2} = \frac{-3 \mp \sqrt{-23}}{4} = -\frac{3}{4} \mp \frac{\sqrt{23}}{4}i$$
$$y = e^{-\frac{3}{4}x}\left(A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right)$$

2.2 Initial Value and Boundary Value Problems

Example: Find the special solution of the equation: $4\frac{d^2y}{dx^2}+16\frac{dy}{dx}+17y=0$

If y = 1 at x = 0 and y = 0 at x = π ?

Solution:

$$4m^2+16m+17=0 \quad \Rightarrow \quad m_{1,2}=\frac{-16\mp\sqrt{256-272}}{8}=\frac{-16\mp\sqrt{-16}}{8}=\frac{-16\mp 4i}{8}=-2\mp\frac{1}{2}i$$

$$y=e^{-2x}(A\cos\frac{x}{2}+B\sin\frac{x}{2}) \quad \Leftrightarrow \textit{the general solution}$$

$$at\ x=0 \quad y=1 \quad \Rightarrow \quad 1=e^0(A\cos 0+B\sin 0) \quad \Rightarrow A=1$$

$$at\ x=\pi \quad y=0 \quad \Rightarrow \quad 0=e^{-2\pi}(1\times 0+B\times 1) \quad \Rightarrow \quad B=0$$

$$\therefore y=e^{-2x}(\cos\frac{x}{2})$$

2.3 Solutions of Nonhomogeneous Linear D.E with constant coefficients:

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots\dots\dots\dots+a_n\ y=f(x)$$

The general solution to the linear nonhomogeneous differential equation is,

$y=y_h+y_p$ where y_p denotes the **particular solution** and y_h associates the **homogeneous solution**.

Two methods can be used for solving Second-order linear nonhomogeneous differential equations with constant coefficient.

2.3.1 The Method of Undetermined Coefficients:

The method is initiated by assuming a particular solution of the form:

$$y_p(x)=A_1y_1(x)+A_2y_2(x)+\dots..+A_ny_n(x)$$

where $A_1, A_2, \dots, A_n$ denote arbitrary multiplicative constants. These constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.