Use the following table to find $y_p(x)$:

f(x)	***$y_p(x)$***
a	A
$a\,x^n$	$A_n x^n + A_{n-1} x^{n-1} + ... + A_1 x + A_o$
$a\,e^{p\,x}$	$A\,e^{p\,x}$
$a\cos\beta x$ $a\sin\beta x$	$A\cos\beta x + B\sin\beta x$
$a\,e^{p\,x}\cos\beta x$ $a\,e^{p\,x}\sin\beta x$	$A\,e^{p\,x}(B_1\cos\beta x + B_2\sin\beta x)$
$a\,x^n e^{px}\cos\beta x$ $a\,x^n e^{px}\sin\beta x$	$A\,e^{px}(A_1 x^n + A_2 x^{n-1} + ... + A_n)(B_1\cos\beta x + B_2\sin\beta x)$

Example: Using the table to write y_p ?

1) $f(x) = 2x^3 \quad \Rightarrow \quad y_p = Ax^3 + Bx^2 + Cx + D$
2) $f(x) = 3e^{2x} \quad \Rightarrow \quad y_p = C\,e^{2x}$
3) $f(x) = \dfrac{1}{2}\sin 2x \quad \Rightarrow \quad y_p = A\cos 2x + B\sin 2x$
4) $f(x) = \sin(x) + \cos(x) \quad \Rightarrow \quad y_p = A\cos x + B\sin x$

Example-1: solve the following D.E? $y''-4y=8x^2$

Solution: $y=y_h+y_p$

$to\ find\ y_h$

$Let\quad y''-4y=0 \quad\Rightarrow\quad \text{homogeneous}$

$m^2-4=0 \quad\Rightarrow\quad m_1=-2 \quad m_2=2$

$y_h=C_1e^{-2x}+C_2e^{2x}$

$to\ find\ y_p \quad\Leftrightarrow\quad \text{since}\quad f(x)=8x^2$

$y_p=Ax^2+Bx+C$

$y'_p=2Ax+B$

$y''_p=2A$

Substitute in the original equation

$2A-4Ax^2-4Bx-4C=8x^2$

$-4Ax^2-4Bx-(4C-2A)=8x^2$

and by equalizing the constants of the two sides of the equation $\Rightarrow$

$-4A=8 \quad\Rightarrow\quad A=-2$

$-4B=0 \quad\Rightarrow\quad B=0$

$-4C+2A=0 \quad\Rightarrow\quad -4C+(2\times(-2))=0 \quad\Rightarrow\quad C=-1$

$\therefore y_p=-2x^2-1$

$y=y_p+y_h \quad\Rightarrow\quad y=C_1e^{-2x}+C_2e^{2x}-2x^2-1$

Example-2: solve the following D.E? $y''+y'-6y=2e^{-2x}$

Solution: $y=y_h+y_p$

$to\ find\ y_p$

$Let\ y''+y'-6y=0 \Rightarrow \quad homogeneous$

$m^2+m-6=0$

$$m_{1,2}=\frac{-1\mp\sqrt{1+24}}{2} \Rightarrow m_1=-3\ \ and\ \ m_2=2$$

$y_h=C_1e^{-3x}+C_2e^{2x}$

$to\ find\ y_p$

$y_p=Ce^{-2x}$

$y'_p=-2Ce^{-2x}$

$y''_p = 4Ce^{-2x}$

substitute in original equation

$4Ce^{-2x} - 2Ce^{-2x} - 6Ce^{-2x} = 2e^{-2x}$

$-4Ce^{-2x} = 2e^{-2x}$

$-4C = 2 \quad \Rightarrow \quad C = -\frac{1}{2}$

$\therefore y_p = -\frac{1}{2}e^{-2x}$

$y = y_h + y_p$

$y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{2}e^{-2x}$

Example-3: solve the following D.E? $y'' - y' - 2y = 10\cos x$

Solution: $y = y_h + y_p$

to find y_h

Let $y'' - y' - 2y = 0 \quad \Rightarrow \quad$ homogeneous

$m^2 - m - 2 = 0 \quad \Rightarrow \quad (m+1)(m-2) = 0 \quad \Rightarrow \quad m_1 = -1 \quad m_2 = 2$

$y_h = C_1 e^{-x} + C_2 e^{2x}$

to find y_p

$y_p = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

substitute in the original equation

$-A\cos x - B\sin x + A\sin x - B\cos x - 2A\cos x - 2B\sin x = 10\cos x$

$(-3A - B)\cos x + (A - 3B)\sin x = 10\cos x$

$-3A - B = 10 \quad$ ....(1)

$A - 3B = 0 \quad$ ....(2) $\quad$ *multiply by* 3 *and add from* 1

$-3A - B = 10$

$3A - 9B = 0$

$-10B = 10 \quad \Rightarrow B = -1 \quad$ *substitute in eq.* (1)

$-3A + 1 = 10 \quad \Rightarrow \quad A = -3$

$y_p = -3\cos x - \sin x$

$y = C_1 e^{-x} + C_2 e^{2x} - 3\cos x - \sin x$

Example-4: solve the following D.E? $y''-3y'+2y=e^x$

Solution: $y=y_h+y_p$

to find y_h

Let $y''-3y'+2y=0 \Rightarrow$ homogeneous

$m^2-3m+2=0 \Rightarrow (m-1)(m-2)=0 \Rightarrow m_1=1 \quad m_2=2$

$y_h=C_1e^x+C_2e^{2x}$

to find y_p

$y_p=Ce^x \otimes$ *but there is a term in the eq. of* y_h *which is also* Ce^x *so we multiply* y_p *by* x

$\therefore \quad y_p=Cxe^x$

$y'_p=C\,xe^x+Ce^x=C(e^x+xe^x)$

$y''_p=C\,xe^x+Ce^x+Ce^x=C(2e^x+xe^x)$

substitute in the original equation

$C(2+x)e^x-3C(1+x)e^x+2Cx\,e^x=e^x$

$2Ce^x+Cx\,e^x-3Ce^x-3Cx\,e^x+2Cx\,e^x=e^x$

$-Ce^x=e^x \Rightarrow C=-1$

$\therefore \quad y_p=-x\,e^x \quad \Rightarrow y=C_1e^x+C_2\,e^{2x}-x\,e^x$