2.3.2 Method of Variation of Parameters:

In general, if *f(x)* is not one of the types of functions considered in the (undetermined coefficients method), or if the differential equation ***does not have constant coefficient***, then this method is preferred.

Variation of parameters is another method for finding a particular solution of the second-order linear differential equation. It can be applied to all linear D.E's. It is therefore more powerful than the undetermined coefficients which is restricted to linear D.E's with constant coefficients and particular forms of *f(x)*.

The general form of the second-order linear D.E is

$$ay'' + by' + cy = f(x) \qquad (1)$$

The solution as we know is $y = y_h + y_p$ where y_h is the general solution of the corresponding homogeneous equation *f(x)*=0 which is expressed as:

$$ay'' + by' + cy = 0$$

and y_p is the particular solution, which can be expressed in this case as:

$$y_p = v_1 y_1 + v_2 y_2$$

The general method for finding a particular solution of the nonhomogeneous equation (1) above, once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring that the

$$v'_1 y_1 + v'_2 y_2 = 0$$

Then we have,

$$v_1' y_1' + v_2' y_2' = f(x)$$

Now, for the unknown functions v'_1 and v'_2, The usual procedure for solving this simple system is to use the *method of determinants* (also known as *Cramer Rule)*

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \times y_2' - y_2 \times y_1'$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = 0 \times y_2' - y_2 \times f(x)$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1 \times f(x) - 0 \times y_1'$$

then,

$v_1{}' = \frac{w_1}{w}$ and $v_2{}' = \frac{w_2}{w}$

after that,

$$v_1(x) = \int v_1{}'\,dx$$

$$v_2(x) = \int v_2{}'\,dx$$

finally,

$$y_p = v_1 y_1 + v_2 y_2$$

and the general solution is:

$$y = y_h + y_p$$

Example-1: Find the general solution to the equation

$$y'' + y = \tan x$$

Solution: The solution of the homogeneous equation

$$y'' + y = 0$$

$$y_h = A\cos x + B\sin x$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$ then,

$$v'_1 y_1 + v'_2 y_2 = 0 \qquad \Rightarrow \qquad v_1{}'\cos x + v_2{}'\sin x = 0$$

$$v_1' \, y_1{}' + v_2' \, y_2{}' = f(x) \qquad \Rightarrow \qquad -v_1{}'\sin x + v_2{}'\cos x = \tan x$$

then,

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 \times y_2' - y_2 \times y_1{}'$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos x \times \cos x + \sin x \times \sin x = \cos^2 x + \sin^2 x$$

$$w_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \times \tan x$$

$$w_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \times \tan x$$

then,

$$v_1' = \frac{w_1}{w} = \frac{-\sin x \times \tan x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}$$

$$v_2' = \frac{w_2}{w} = \frac{\cos x \times \tan x}{\cos^2 x + \sin^2 x} = \sin x$$

after that,

$$v_1(x) = \int v_1{}'\,dx = \int \frac{-\sin^2 x}{\cos x}dx = -\int (\sec x - \cos x)\,dx$$

$$v_1(x) = -\ln(\sec x + \tan x) + \sin x$$

$$v_2(x) = \int v_2{}'\,dx = \int \sin x\,dx = -\cos x$$

Finally,

$$y_p = v_1 y_1 + v_2 y_2 = [-\ln(\sec x + \tan x) + \sin x]\cos x + (-\cos x)\sin x$$

$$y_p = (-\cos x)\ln(\sec x + \tan x)$$

$$y = y_h + y_p = A\cos x + B\sin x - (\cos x)\ln(\sec x + \tan x)$$

Example2: $y'' + y' - 2y = xe^x$

Solution:

$$m^2 + m - 2 = 0$$

$$y_h = c_1 e^{-2x} + c_2 e^x$$

Since $y_1(x) = e^{-2x}$ and $y_2(x) = e^x$ then,

$$v'_1 y_1 + v'_2 y_2 = 0 \qquad \Rightarrow \qquad v_1{}'e^{-2x} + v_2{}'e^x = 0$$

$$v_1' y_1{}' + v_2' y_2{}' = f(x) \qquad \Rightarrow \qquad -2v_1{}'e^{-2x} + v_2{}'e^x = xe^x$$

then,

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 \times y_2' - y_2 \times y_1{}'$$

$$w = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = 3e^{-x}$$

$$w_1 = \begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix} = -xe^{2x}$$

$$w_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix} = xe^{-x}$$

then,

$$v_1' = \frac{w_1}{w} = \frac{-xe^{2x}}{3e^{-x}} = \frac{-1}{3}xe^{3x}$$

$$v_2' = \frac{w_2}{w} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}$$

after that,

$$v_1(x) = \int v_1'\,dx = \int \frac{-1}{3} x e^{3x} dx = -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3} dx\right)$$

$$v_1(x) = \frac{1}{27}(1-3x)e^{3x}$$

$$v_2(x) = \int v_2'\,dx = \int \frac{x}{3} dx = \frac{x^2}{6}$$

Finally,

$$y_p = v_1 y_1 + v_2 y_2 = \left[\frac{(1-3x)e^{3x}}{27}\right] e^{-2x} + \left(\frac{x^2}{6}\right) e^x$$

$$y_p = \frac{1}{27}e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2 e^x$$

$$\therefore y = y_h + y_p = c_1 e^{-2x} + c_2 e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2 e^x$$

where the term *(1/27)*e^xin y_p has been absorbed into the term $C_2 e^x$ in the complementary solution.