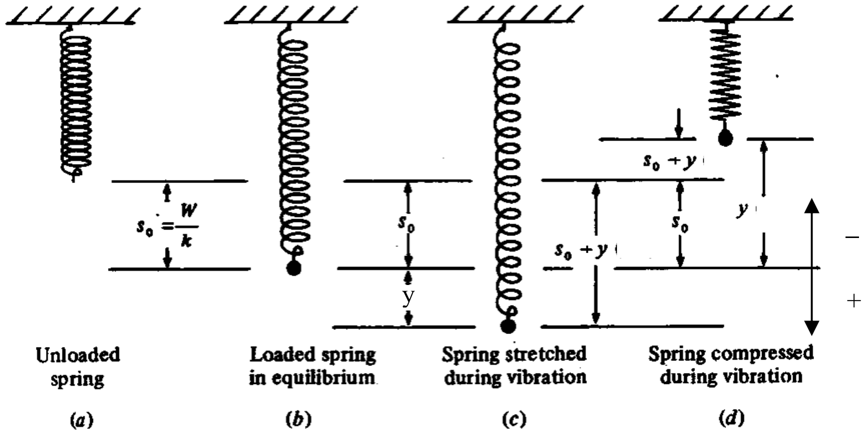
Applications of Second Order Linear Differential Equations with constant coefficients:

Free Oscillation:



Static Case:

$$\sum fy = 0$$

$$mg - ks_o = 0$$

Dynamic Case:

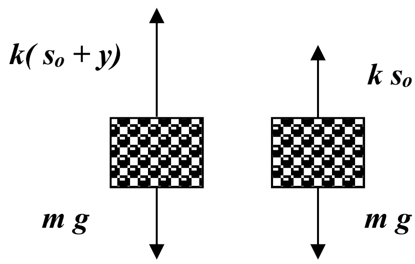
$$F = m\frac{d^2y}{dt^2}$$

$$mg - k(s_o + y) = my''$$

$$but\ mg = ks_o \quad \Rightarrow$$

$$-ky = my'' \quad \Rightarrow$$

$$\boxed{my'' + ky = 0}$$



$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0 \ where$$

$$\omega^2 = \frac{k}{m} \quad or \quad \omega = \sqrt{\frac{k}{m}}$$

$$m^2 + \omega^2 = 0$$

$$m^2 = -\omega^2$$

$$m = \mp \omega i$$

$$\boxed{y = A\cos\omega t + B\sin\omega t}$$

Case I (General case):

$y(0) = y_o \quad \Leftrightarrow \quad y'(0) = 0$

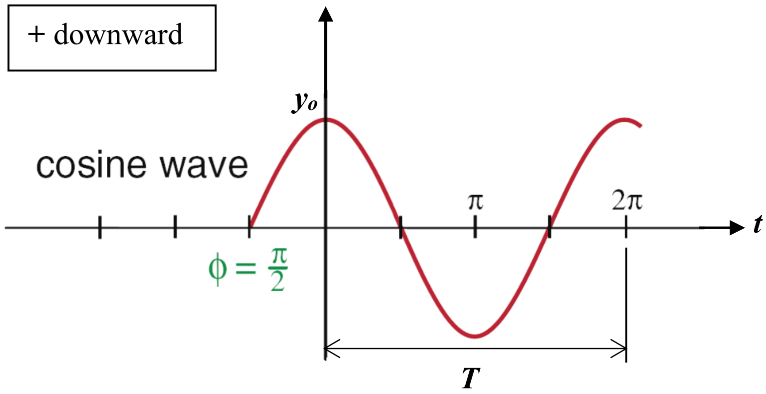
$A = y_o \quad B = 0 \quad \Rightarrow$

$$\boxed{y = y_o \cos\omega t}$$

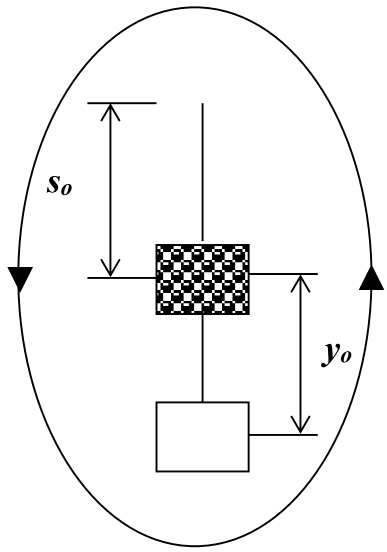
for a complete cycle

$\omega t = 2\pi \quad \Rightarrow \quad t = \frac{2\pi}{\omega}$

$\therefore T = \frac{2\pi}{\omega}$ *cycle period*



$\cos\phi = \cos(\phi + 2\pi) \quad \Rightarrow \quad \cos\omega t = \cos(\omega t + 2\pi)$



Complete Cycle

Case II:

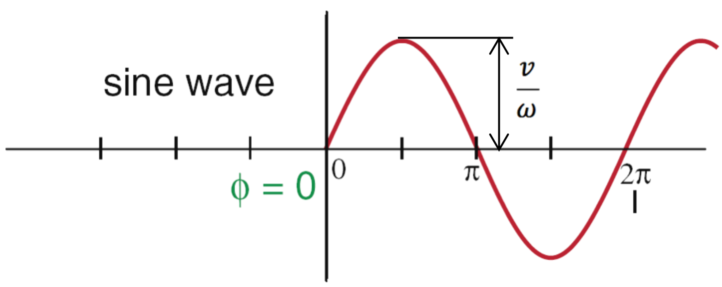
$y(0) = 0 \quad \Leftrightarrow \quad y'(0) = v$

$0 = A + 0 \quad \Rightarrow \quad A = 0$

$\frac{dy}{dt} = \omega B \cos\omega t$

$\left.\frac{dy}{dt}\right|_{t=o} = \omega B = v$

$$\boxed{y = \frac{v}{\omega}\sin\omega t}$$



$$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T \propto \frac{1}{\omega}$$

Case III:

$$y(0) = y_o \qquad y'(0) = v$$

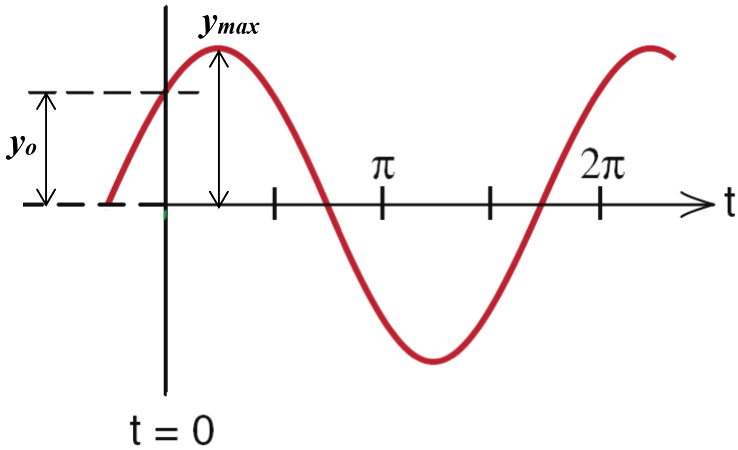
$$y_o = A + 0 \quad \Rightarrow \quad A = y_o$$

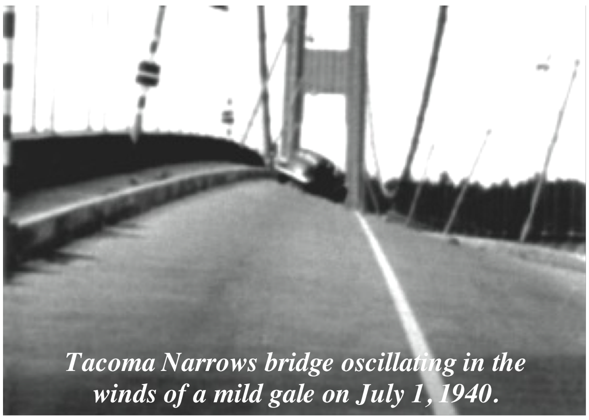
$$\frac{dy}{dt} = -\omega A \sin \omega t + \omega B \cos \omega t$$

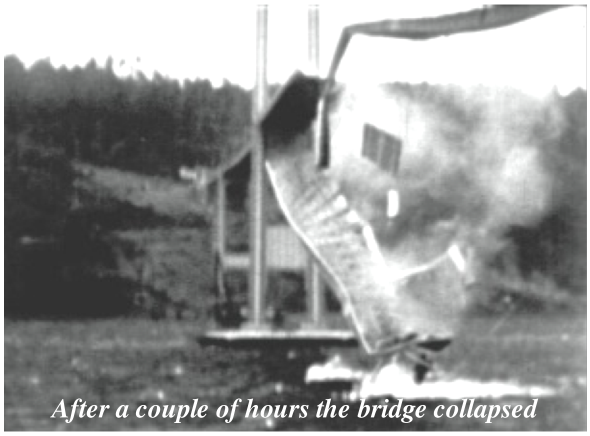
$$v = 0 + \omega B$$

$$B = \frac{v}{w}$$

$$\boxed{y = y_o \cos \omega t + \frac{v}{\omega} \sin \omega t}$$




Tacoma Narrows bridge oscillating in the winds of a mild gale on July 1, 1940.


After a couple of hours the bridge collapsed

Example: A weight of (*7 N*) is suspended from a spring of modulus (***k=36/35 N/cm***). At ***t = 0***, while the weight in static equilibrium it is given suddenly an initial velocity of (***48 cm/sec***) in downward.

a. Find the vertical displacement as a function of ***t***.
b. What are the period and frequency of motion?
c. Through what amplitude does the weight moves?
d. At what time does the load reach its extreme displacement above and below its equilibrium position?

Solution:

a.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{36/35}{7/980}} = 12$$

$$y = A\cos 12t + B\sin 12t$$

$$y(0) = A + 0 \quad \Rightarrow \quad A = 0$$

$$y = B\sin 12t$$

$$y' = 12\,B\cos 12t$$

$$y'\big|_{t=o} = 12\,B \quad \Rightarrow \quad 48 = 12\,B \quad \Rightarrow \quad B = 4$$

$$\boxed{y = 4\sin 12t}$$

b.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{12} = \frac{\pi}{6} \quad \sec/cycle$$

$$f = \frac{6}{\pi} \quad cycle/\sec \quad (Hertz)$$

c.

$$y = 4\sin 12\,t \qquad -1 \le \sin 12\,t \le 1$$

$$y_{\max} = 4 \qquad y_{\min} = -4$$

$$Amplitude = 4 + |-4| = 8$$

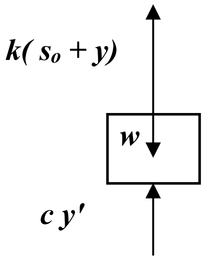
d.

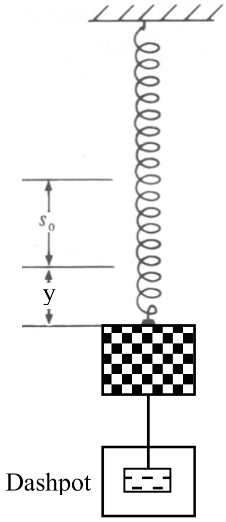
$$\sin 12\,t = \mp 1$$

$$12t = (2n+1)\frac{\pi}{2} \quad \Leftrightarrow \quad t = \left(\frac{1+2n}{24}\right)\pi \qquad n = 0, 1, 2, \quad Multiplication\ of\ \frac{\pi}{2}$$

Damped Oscilation:

Damping $\propto \frac{dy}{dt} = c\,y'$





$$\oplus \downarrow \sum f\,y = M\,y''$$

$$w - k(s_o + y) - c\,y' = M\,y''$$

$$k\,y + M\,y'' + cy' = 0$$

$$\boxed{M\,y'' + c\,y' + k\,y = 0}$$ Arithmetic Model

$M\,m^2 + c\,m + k = 0$ Characteristic Equation

$$m_{1,2} = \frac{-c \mp \sqrt{c^2 - 4\,M\,k}}{2\,M} = \frac{-c}{2\,M} \mp \frac{1}{2\,M}\sqrt{c^2 - 4\,M\,k}$$

$$m_{1,2} = -\alpha \mp \beta$$

$$\alpha = \frac{c}{2\,M} \qquad \beta = \frac{1}{2\,M}\sqrt{c^2 - 4\,M\,k}$$

Critical Damping Coefficient:

The Critical Damping Coefficient is the value of ***c*** which makes:

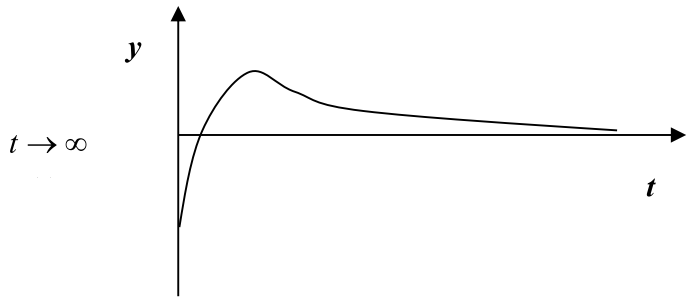
$$\sqrt{c^2 - 4\,M\,k} = 0$$

$$\boxed{C_{cr} = 2\sqrt{k\,M}}$$

Tacoma Narrows bridge in an n = 2 second harmonic standing wave pattern.

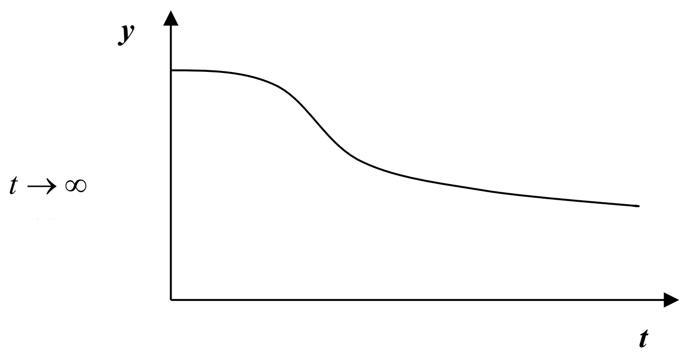
Case 1: $c \succ C_{cr}$ *i.e* $\sqrt{c^2 - 4Mk} \succ 0 \Rightarrow$ *Over Damping* $\Leftrightarrow$ α *and* β *are real*

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$



Case 2: $c = C_{cr}$ *i.e* $m_1 = m_2 = -\alpha \Rightarrow$ *Ciritical Damping*

$$y(t) = (c_1 + c_2\,t)\,e^{-\alpha t}$$



Case 3: $c \prec C_{cr} \Leftrightarrow$ *Under Damping*

$$m_{1,2} = \frac{-c}{2M} \mp \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

$$= -\alpha \mp \beta$$

$$\alpha = \frac{c}{2M} \quad \beta = \frac{1}{2M}\sqrt{c^2 - 4Mk}$$

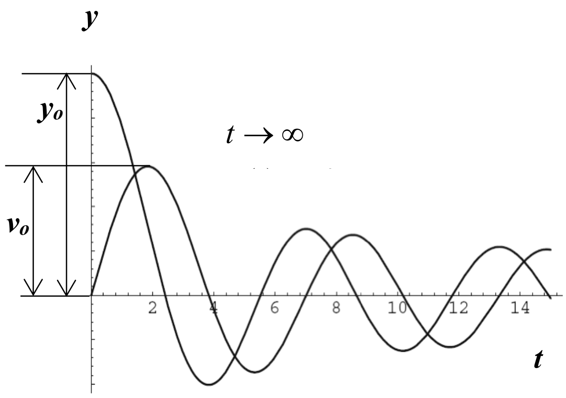
$$= \frac{1}{2M}\sqrt{4Mk - c^2} \times i$$

$$= \omega^* i$$

$$m_{1,2} = -\alpha \mp \omega^* i$$

$$y(t) = e^{pt}(A\cos qt + B\sin qt)$$

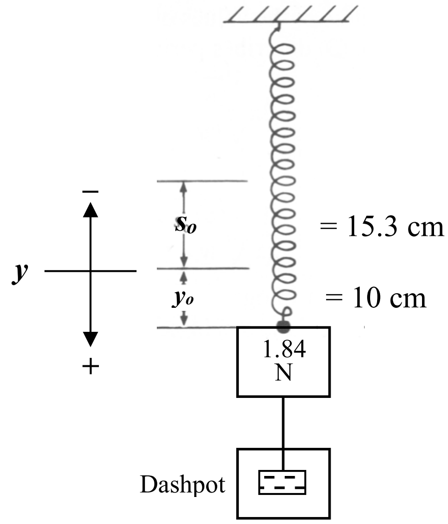
$$y = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$



Example: A ***(1.84 N)*** body is suspended by a spring which is stretched ***(15.3 cm)*** when it is loaded. If the body is drawn down ***(10 cm)*** from the position of equilibrium; find the position of the spring as a function of time ***(t)*** if:

1. c = 1.5
2. c = 3.75
3. c = 3

Solution:



$$M\,y''+c\,y'+k\,y=0$$

$$M=\frac{w}{g}=\frac{1.84}{9.8}=0.188\ kg$$

$$k=\frac{w}{s_o}=\frac{1.84}{0.153}=12\ \frac{N}{m}$$

1) for c = 1.5

$$0.188\,y''+1.5\,y'+12\,y=0$$

$$y''+8\,y'+64\,y=0$$

$$m^2+8m+64=0$$

$$m_{1,2}=\frac{-8\mp\sqrt{64-4\times 64}}{2}=-4\mp 4\sqrt{3}\,i \quad\Leftrightarrow\quad \textit{Under Damping}$$

$$y(t)=e^{-4t}(A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t)$$

$\underline{\textit{Initial Conditions}}:$

$$at \quad t=0 \quad\Rightarrow\quad y'=o \quad\Rightarrow y=0.1m$$

$$y(0)=A+0=0.1 \quad\Rightarrow\quad A=0.1$$

$$y'(t)=\left[-4\sqrt{3}\,A\sin 4\sqrt{3}\,t+4\sqrt{3}\,B\cos 4\sqrt{3}\,t\right]+\left[A\cos 4\sqrt{3}\,t+B\sin 4\sqrt{3}\,t\right](-4e^{-4t})$$

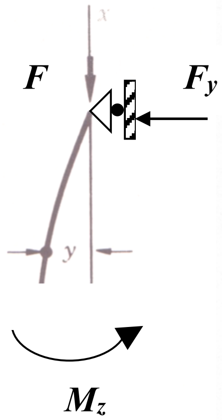
$$y'(0)=4\sqrt{3}\,B-4A=4\sqrt{3}\,B-4\times 0.1=0 \quad\Rightarrow\quad B=\frac{1}{10\sqrt{3}}$$

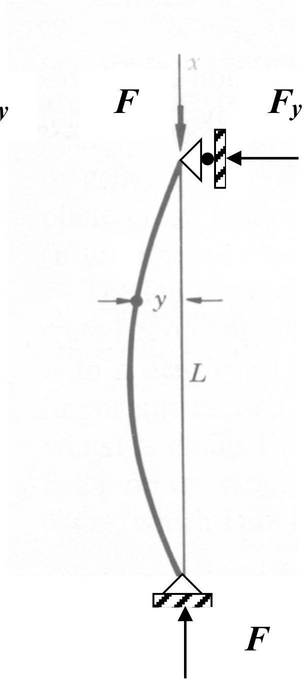
$$y=e^{-4t}(0.1\cos 4\sqrt{3}\,t+\frac{1}{10\sqrt{3}}\sin 4\sqrt{3}\,t)$$

2) for c = 3.75

3) for c = 3

Column Buckling:





$$\sum f_y = 0 \quad \Rightarrow \quad F_y = 0$$

$$M = -F\,y$$

$$\frac{d^2 y}{d x^2} = \frac{M}{E\,I}$$

$$y'' = -\frac{F}{E\,I} y \quad \Rightarrow \quad y'' + \frac{F}{E\,I} y = 0$$

$$m^2 + \frac{F}{E\,I} = 0 \quad \Rightarrow \quad m^2 = -\frac{F}{E\,I}$$

$$m_{1,2} = \mp \sqrt{\frac{F}{E\,I}}\, i$$

$$y = A\cos\sqrt{\frac{F}{E\,I}}\,x + B\sin\sqrt{\frac{F}{E\,I}}\,x$$

$$at \quad x = 0 \qquad y = 0$$

$$0 = A + B \times 0 \quad \Rightarrow \quad A = 0$$

$$\therefore\ y = B\sin\sqrt{\frac{F}{E\,I}}\,x$$

$$at \quad x = L \qquad y = o \quad also$$

but this means that $B = 0$ *which results in zero equation. However,*

If the load F *have just the right value to make:*

$\sqrt{\frac{F}{E\,I}}\,L = n\,\pi$ *then the last equation will be satisfied without* B *being* 0 *and*

the equilibrium is possible in a deflected position defined by:

$$\sqrt{\frac{F}{E\,I}} = \frac{n\,\pi}{L} \qquad n = 1, 2, 3, \ \ldots\ldots\ldots \quad \Rightarrow \quad y = B\sin\frac{n\,\pi\,x}{L}$$

$$\sqrt{\frac{F_n}{E\,I}} = \frac{n\,\pi}{L} \quad \Rightarrow \quad \frac{F_n}{E\,I} = \left(\frac{n\,\pi}{L}\right)^2 \quad \Rightarrow \quad F_n = \left(\frac{n\,\pi}{L}\right)^2 E\,I$$

$$for\ n = 1 \quad \Rightarrow \quad F_1 = \frac{\pi^2\,E\,I}{L^2}$$

For values of F below the lowest critical load, the column will remain in its undeflected vertical position, or if displaced slightly from it, will return to it as an equilibrium configuration. For values of F above the lowest critical load and different from the higher critical loads, the column can theoretically remain in a vertical position, but the equilibrium is unstable, and if the column is deflected slightly, it will not return to a vertical position but will continue to deflect until it collapses. Thus, only the lowest critical load is of much practical significance.