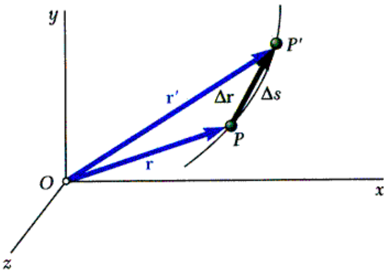
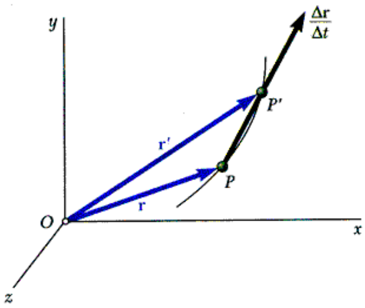
## *1-7 Curvilinear Motion: Position, Velocity & Acceleration*

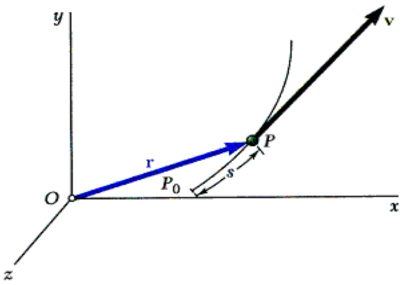



The position vector of a particle at time t is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.
Consider a particle which occupies position *P* defined by at time *t* and *P'* defined by at *t* + *Δt*, then:

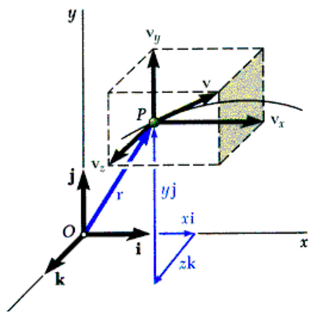
Instantaneous velocity (vector) & Instantaneous speed (scalar)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$




## *1-8 Rectangular Components of Velocity & Acceleration*

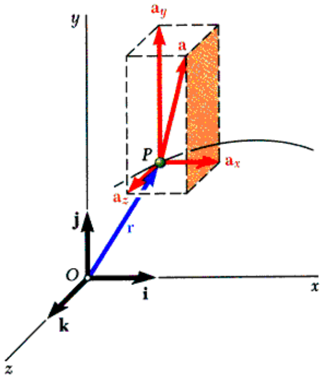



When position vector of particle *P* is given by its rectangular components, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
Velocity vector,

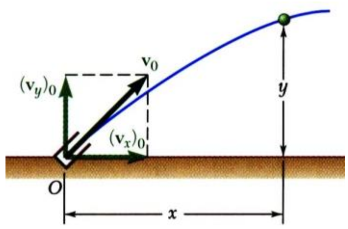
$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$
$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$



- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

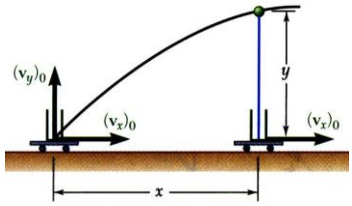



$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

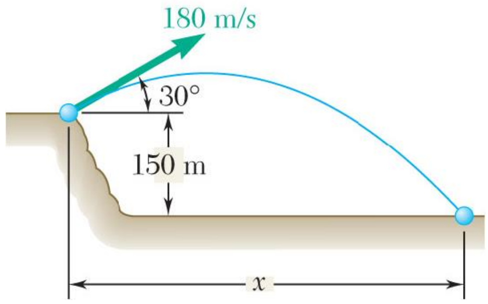
with initial conditions: $x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0$

Integrating twice yields:
$$\begin{array}{lll} v_x = (v_x)_0 & v_y = (v_y)_0 - gt & v_z = 0 \\ x = (v_x)_0 t & y = (v_y)_0 y - \frac{1}{2}gt^2 & z = 0 \end{array}$$

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.




**Example 4:** A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30°with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

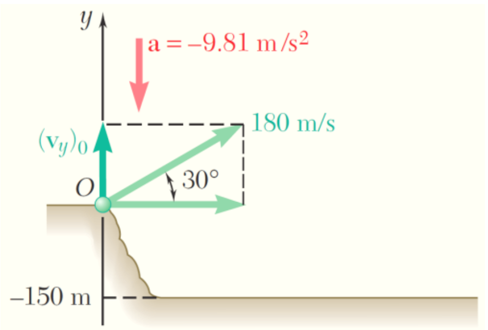



SOLUTION:

- Maximum elevation occurs when $v_y=0$, the Vertical motion is uniformly accelerated, then:

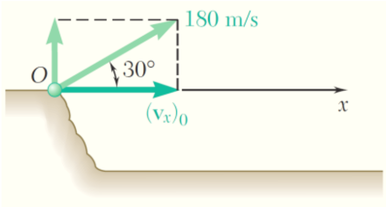
$$(v_y)_0 = (180\text{ m/s})\sin 30° = +90\text{ m/s}$$

$$\begin{array}{llr} v_y = (v_y)_0 + at & v_y = 90 - 9.81t & (1) \\ y = (v_y)_0 t + \frac{1}{2}at^2 & y = 90t - 4.90t^2 & (2) \\ v_y^2 = (v_y)_0^2 + 2ay & v_y^2 = 8100 - 19.62y & (3) \end{array}$$



Also horizontal motion – uniformly accelerated, then: $(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$

$$x = (v_x)_0 t \qquad x = 155.9t \quad \ldots\ldots(4)$$




$$y = -150 \text{ m}$$

$$-150 = 90t - 4.90t^2 \qquad t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

Projectile strikes the ground at: Substitute into equation (1)

⇨

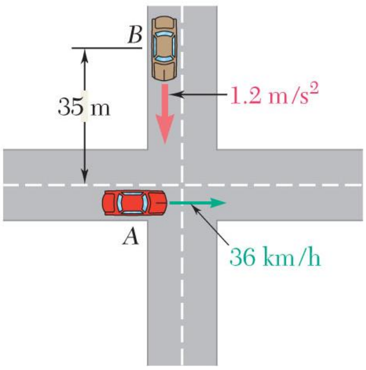
Substitute *t* into equation (4): $x = 155.9(19.91)$ ⇨ $x = 3100 \text{ m}$

Maximum elevation occurs when $v_y = 0$

$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

Maximum elevation above the ground = $150 \text{ m} + 413 \text{ m} = 563 \text{ m}$

**Example 4:** Automobile *A* is traveling east at the constant speed of 36 km/h. As automobile *A* crosses the intersection shown, automobile *B* starts from rest 35 m north of the intersection and moves south with a constant acceleration of 1.2 m/s$^2$. Determine the position, velocity, and acceleration of *B* relative to *A* 5 *s* after *A* crosses the intersection.

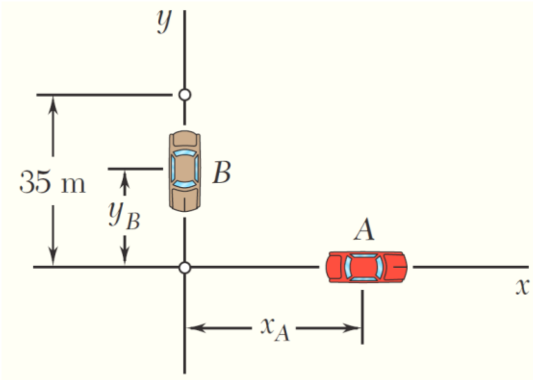



SOLUTION:

Given: $v_A$=36 km/h, $a_A$= 0, $(x_A)0$ = 0, $(v_B)0$= 0, $a_B$= - 1.2 m/s$^2$, $(y_A)0$ = 35 m

- Determine motion of Automobile A:

$$v_A = \left(36\frac{\text{km}}{\text{h}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 10 \text{ m/s}$$




We have uniform motion for A so:

$$a_A = 0$$
$$v_A = +10 \text{ m/s}$$
$$x_A = (x_A)_0 + v_A t = 0 + 10t$$

At t = 5 s:

$$a_A = 0$$
$$v_A = +10 \text{ m/s}$$
$$x_A = +(10 \text{ m/s})(5 \text{ s}) = +50 \text{ m}$$

$$\mathbf{a}_A = 0$$
$$\mathbf{v}_A = 10 \text{ m/s} \rightarrow$$
$$\mathbf{r}_A = 50 \text{ m} \rightarrow$$

- Determine motion of Automobile B:

We have uniform acceleration for B so:

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2\,t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \tfrac{1}{2}a_B t^2 = 35 + 0 - \tfrac{1}{2}(1.2)t^2$$

At $t$ = 5 s:

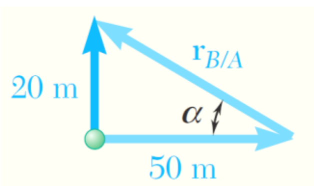
$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s}$$
$$y_B = 35 - \tfrac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = +20 \text{ m}$$

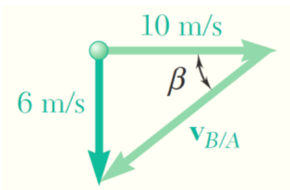
$$\mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$\mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$\mathbf{r}_B = 20 \text{ m} \uparrow$$

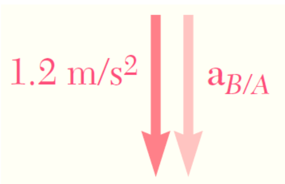
Since :

$$\mathbf{a}_A = 0 \qquad \mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$\mathbf{v}_A = 10 \text{ m/s} \rightarrow \qquad \mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$\mathbf{r}_A = 50 \text{ m} \rightarrow \qquad \mathbf{r}_B = 20 \text{ m} \uparrow$$

Then the problems can be solve geometrically, and apply the arctangent relationship:




$r_{B/A} = 53.9$ m $\qquad \alpha = 21.8°$ $\qquad v_{B/A} = 11.66$ m/s $\qquad \beta = 31.0°$

$\mathbf{r}_{B/A} = 53.9$ m ⦣ 21.8° $\qquad \mathbf{v}_{B/A} = 11.66$ m/s ⦤ 31.0° $\qquad \mathbf{a}_{B/A} = 1.2 \text{ m/s}^2 \downarrow$

Or one can solve the problems using vectors to obtain equivalent results:

$$\mathbf{r_B} = \mathbf{r_A} + \mathbf{r_{B/A}} \qquad \mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}} \qquad \mathbf{a_B} = \mathbf{a_A} + \mathbf{a_{B/A}}$$

$$20\mathbf{j} = 50\mathbf{i} + \mathbf{r_{B/A}} \qquad -6\mathbf{j} = 10\mathbf{i} + \mathbf{v_{B/A}} \qquad -1.2\mathbf{j} = 0\mathbf{i} + \mathbf{a_{B/A}}$$

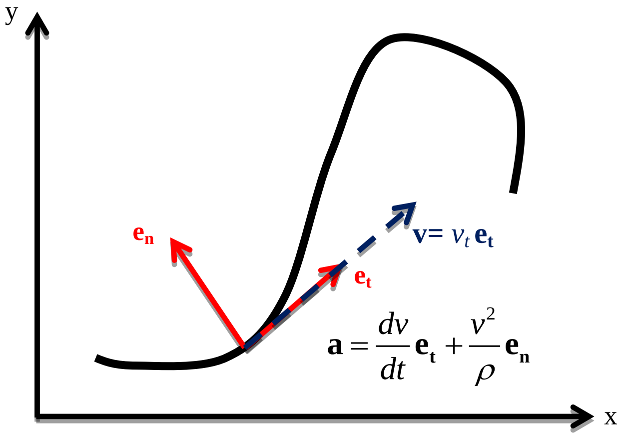
$$\mathbf{r_{B/A}} = 20\mathbf{j} - 50\mathbf{i} \ \text{(m)} \qquad \mathbf{v_{B/A}} = -6\mathbf{j} - 10\mathbf{i} \ \text{(m/s)} \qquad \mathbf{a_{B/A}} = -1.2\mathbf{j} \ (\text{m/s}^2)$$

$$v_{B/A} = 11.66 \text{ m/s}$$

Physically, a rider in car A would "see" car B traveling south and west.

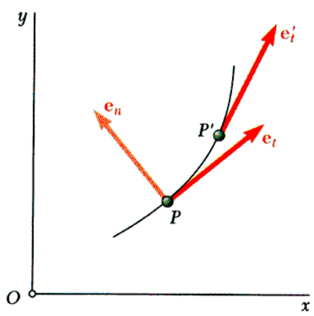
## 1-9 Tangential and Normal Components

If we have an idea of the path of a vehicle, it is often convenient to analyze the motion using tangential and normal components (sometimes called path coordinates).




- The tangential direction ($\mathbf{e_t}$) is tangent to the path of the particle. This velocity vector of a particle is in this direction
- The normal direction ($\mathbf{e_n}$) is perpendicular to **et** and points towards the inside of the curve.
- The acceleration can have components in both the $\mathbf{e_n}$ and $\mathbf{e_t}$ directions

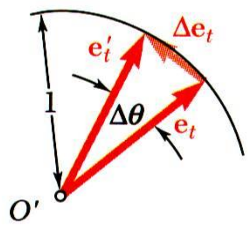
To derive the acceleration vector in tangential and normal components, define the motion of a particle as shown in the figure. $\vec{e}_t$ and $\vec{e}'$ are tangential unit vectors for the particle path at *P* and *P'*. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta\theta$ is the angle between them.




$$\Delta e_t = 2\sin(\Delta\theta/2)$$

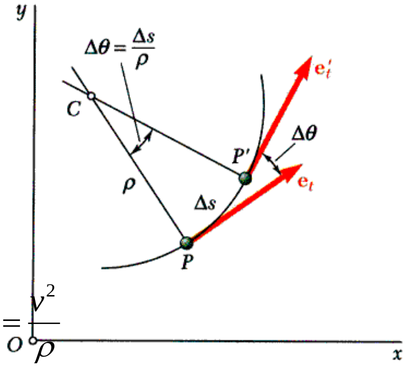
$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$




$\vec{v} = v\vec{e}_t$

With the velocity vector expressed as



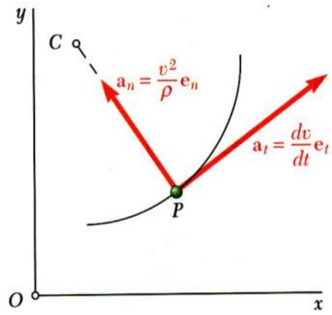

, the particle acceleration may be written as:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

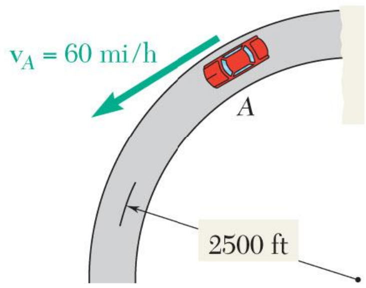
But, $\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$

After substituting, $\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}$

- The tangential component of acceleration reflects change of speed and the normal component reflects change of direction.
- The tangential component may be positive or negative. Normal component always points toward center of path curvature.

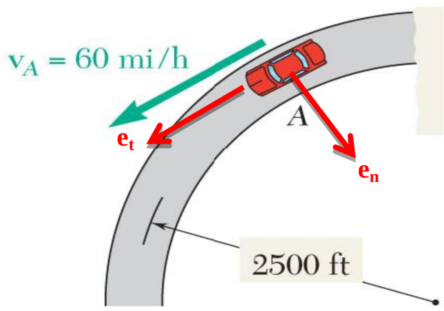



**Example 5:** A motorist is traveling on a curved section of highway of radius 2500 ft at the speed of 60 mi/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 45 mi/h, determine the acceleration of the automobile immediately after the brakes have been applied.




SOLUTION: Define your coordinate system

Then Determine velocity and acceleration in the tangential direction

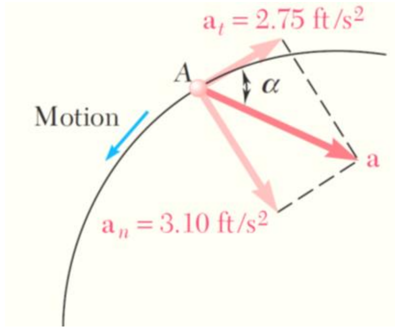



$$60 \text{ mi/h} = \left(60\frac{\text{mi}}{\text{h}}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 88 \text{ ft/s}$$

$$45 \text{ mi/h} = 66 \text{ ft/s}$$

The deceleration constant, therefore;

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{66 \text{ ft/s} - 88 \text{ ft/s}}{8 \text{ s}} = -2.75 \text{ ft/s}^2$$




Immediately after the brakes are applied, the speed is still 88 ft/s

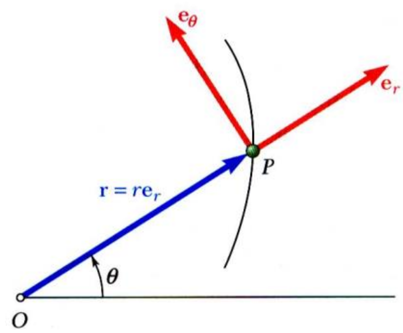
$$a_n = \frac{v^2}{\rho} = \frac{(88\ \text{ft/s})^2}{2500\ \text{ft}} = 3.10\ \text{ft/s}^2$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{2.75^2 + 3.10^2}$$

$$\tan\alpha = \frac{a_n}{a_t} = \frac{3.10\ \text{ft/s}^2}{2.75\ \text{ft/s}^2}$$

$\mathbf{a} = 4.14\ \text{ft/s}^2$ and $\alpha = 48.4°$

## 1-10 Radial and Transverse Components




The position of a particle *P* is expressed as a distance *r* from the origin *O* to *P*– this defines the radial direction $\mathbf{e_r}$. The transverse direction $\mathbf{e_\theta}$ is perpendicular to $\mathbf{e_r}$ : $\vec{r} = r\vec{e}_r$

The particle velocity vector is: $\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

The particle acceleration vector is: $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$

One can derive the velocity and acceleration relationships by recognizing that the unit vectors change direction. The particle velocity vector is:

$$\vec{v} = \frac{d}{dt}\left(r\vec{e}_r\right) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta = \dot{r}\,\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

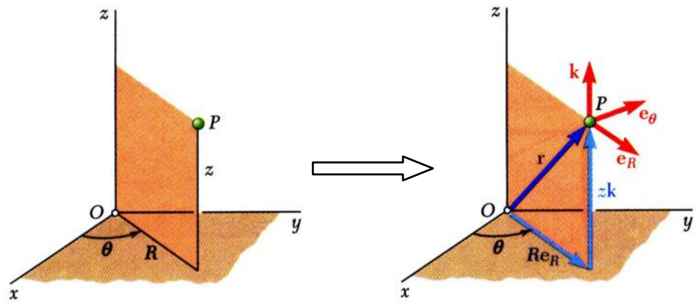
$$\vec{r} = r\vec{e}_r \ldots\ldots\ldots \frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta\frac{d\theta}{dt} \ldots\ldots\ldots\ldots \frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r\frac{d\theta}{dt}$$

Similarly, the particle acceleration vector is:

$$\vec{a}=\frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r+r\frac{d\theta}{dt}\vec{e}_\theta\right)=\frac{d^2r}{dt^2}\vec{e}_r+\frac{dr}{dt}\frac{d\vec{e}_r}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta+r\frac{d^2\theta}{dt^2}\vec{e}_\theta+r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$
$$=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors

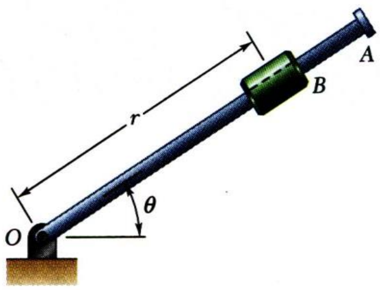



- Position vector: $\vec{r}=R\vec{e}_R+z\vec{k}$
- Velocity vector:

$$\vec{v}=\frac{d\vec{r}}{dt}=\dot{R}\vec{e}_R+R\dot{\theta}\vec{e}_\theta+\dot{z}\vec{k}$$

- Acceleration vector: $\vec{a}=\frac{d\vec{v}}{dt}=\left(\ddot{R}-R\dot{\theta}^2\right)\vec{e}_R+\left(R\ddot{\theta}+2\dot{R}\dot{\theta}\right)\vec{e}_\theta+\ddot{z}\vec{k}$

**Example 6:** Rotation of the arm about O is defined by $\theta$ = 0.15$t^2$ where $\theta$ is in radians and *t* in seconds. Collar B slides along the arm such that $r$ = 0.9 - 0.12$t^2$ where *r* is in meters. After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to the arm.




SOLUTION:

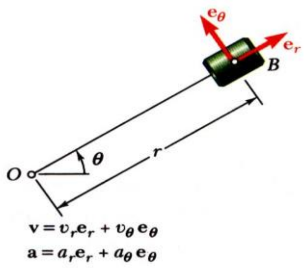
- Evaluate time *t* for $\theta$ = 30°: $\theta=0.15t^2=30°=0.524\,\text{rad}$ $\quad t=1.869\,\text{s}$
- Evaluate radial and angular positions, and first and second derivatives at time *t*.

$r=0.9-0.12t^2=0.481\,\text{m}\quad \dot{r}=-0.24t=-0.449\,\text{m/s}$

.$\ddot{r}=-0.24\,\text{m/s}^2$

$\theta=0.15t^2=0.524\,\text{rad}\qquad \dot{\theta}=0.30t=0.561\,\text{rad/s}$

$\ddot{\theta}=0.30\,\text{rad/s}^2$

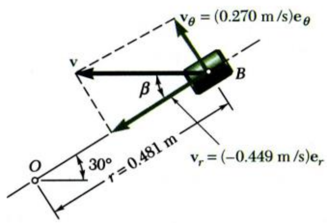



- Calculate velocity and acceleration:

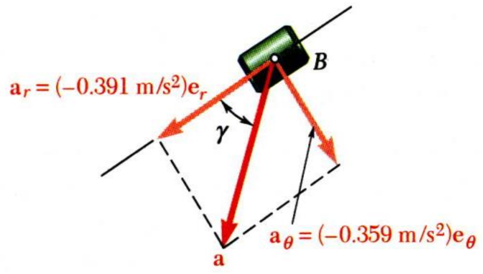
$v_r=\dot{r}=-0.449\,\text{m/s}$

$v_\theta=r\dot{\theta}=(0.481\,\text{m})(0.561\,\text{rad/s})=0.270\,\text{m/s}$

$v=\sqrt{v_r^2+v_\theta^2}=0.524\,\text{m/s}\qquad \beta=\tan^{-1}\frac{v_\theta}{v_r}=31.0°$



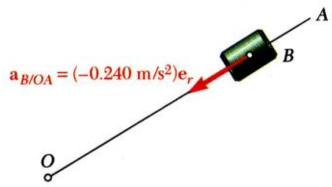
$$a_r = \ddot{r} - r\dot{\theta}^2 = -0.240\,\text{m/s}^2 - (0.481\,\text{m})(0.561\,\text{rad/s})^2$$
$$= -0.391\,\text{m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.481\,\text{m})(0.3\,\text{rad/s}^2) + 2(-0.449\,\text{m/s})(0.561\,\text{rad/s})$$
$$= -0.359\,\text{m/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$



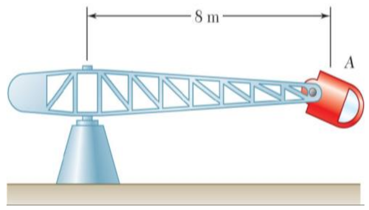

$$a = 0.531\,\text{m/s} \qquad \gamma = 42.6°$$

- Evaluate acceleration with respect to arm. Motion of collar with respect to arm is rectilinear and defined by coordinate *r*.

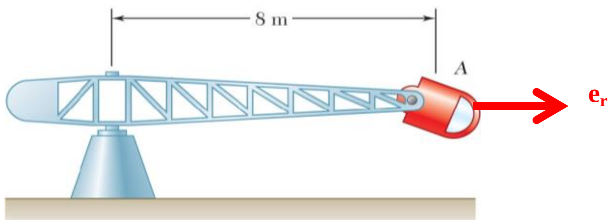
$$a_{B/OA} = \ddot{r} = -0.240\,\text{m/s}^2$$




**Example 7:** The angular acceleration of the centrifuge arm varies according to $\ddot{\theta} = 0.05\theta$ (rad/s$^2$) where θ is measured in radians. If the centrifuge starts from rest, determine the acceleration magnitude after the gondola has traveled two full rotations.



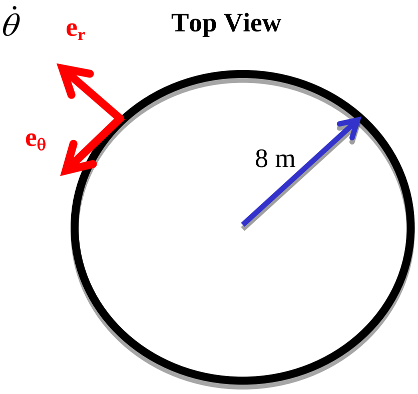

SOLUTION:




- Define your coordinate system
- Determine the angular velocity

$$\ddot{\theta} = 0.05\theta \ \ (\text{rad/s}^2)$$

Acceleration is a function of position, so use: $\ddot{\theta}d\theta = \dot{\theta}d\dot{\theta}$




- Evaluate the integral: $\int_0^{(2)(2\pi)} 0.05\theta d\theta = \int_0^{\dot{\theta}} \dot{\theta} d\dot{\theta}$

$$\left.\frac{0.05\theta^2}{2}\right|_0^{2(2\pi)} = \left.\frac{\dot{\theta}^2}{2}\right|_0^{\dot{\theta}} \qquad \dot{\theta}^2 = 0.05[2(2\pi)]^2$$

- Determine the angular velocity: $\dot{\theta}^2 = 0.05[2(2\pi)]^2$

- Determine the angular acceleration: $\ddot{\theta} = 0.05\theta = 0.05(2)(2\pi) = 0.6283 \text{ rad/s}^2$
- Find the radial and transverse accelerations:

$$\begin{aligned}\vec{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta \\ &= \left(0 - (8)(2.8099)^2\right)\vec{e}_r + \left((8)(0.6283) + 0\right)\vec{e}_\theta \\ &= -63.166\,\vec{e}_r + 5.0265\,\vec{e}_\theta \quad (\text{m/s}^2)\end{aligned}$$

$$a_{mag} = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-63.166)^2 + \left[5.0265\right]^2}$$

$$a_{mag} = 63.365 \text{ m/s}^2$$