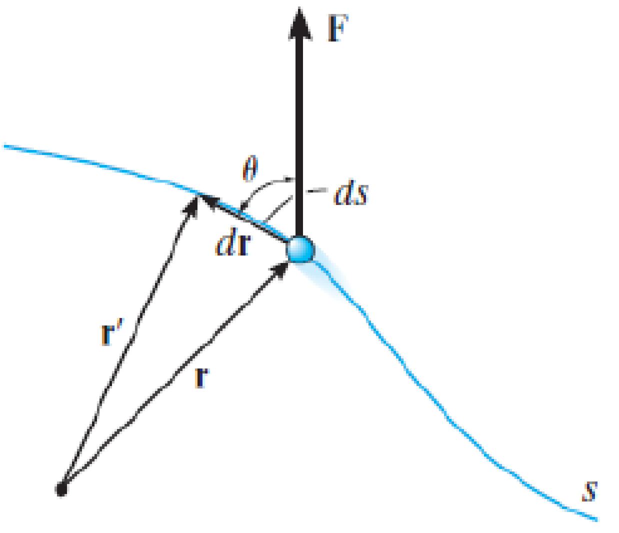
Engineering Mechanics-Dynamics

Chapter Two
Kinetics of Particles

CHAPTER TWO
KINETICS OF PARTICLES: WORK AND ENERGY

2-9 *The Work of a Force*



A force F will do work on a particle only when the particle undergoes a displacement in the direction of the force. For example, if the force F in Fig. causes the particle to move along the path s from position r to a new position r', the displacement is then $dr = r' - r$. The magnitude of dr is ds, the length of the differential segment along the path. If the angle between the tails of dr and F is θ, then the work done by F is a scalar quantity, defined by

$$dU = F\, ds \cos\theta$$

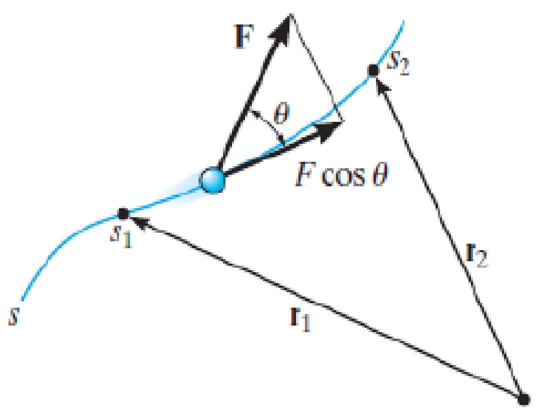
By definition of the dot product this equation can also be written as

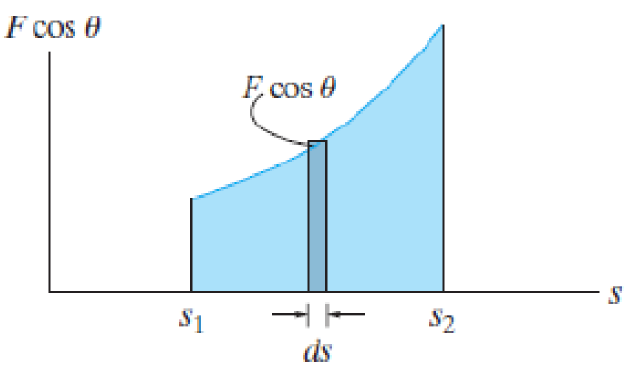
$$dU = F \cdot dr$$

2-9-1 *Work of a Variable Force*

If the particle acted upon by the force F undergoes a finite displacement along its path from r_1 to r_2 or s_1 to s_2 , the work of force F is determined by integration. Provided F and θ can be expressed as a function of position, then

$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F \cos\theta \, ds$$





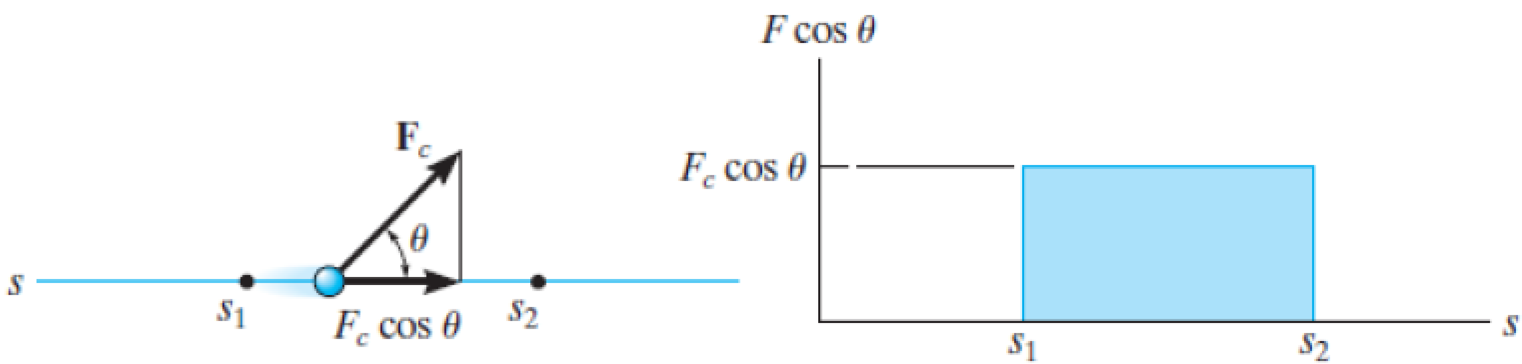
2-9-2 ***Work of a Constant Force Moving Along a Straight Line***

If the force $\boldsymbol{F_c}$ has a constant magnitude and acts at a constant angle θ from its straight-line path, then the component of $\boldsymbol{F_c}$ in the direction of displacement is always $\boldsymbol{F_c}$ cos θ. The work done by $\boldsymbol{F_c}$ when the particle is displaced from s_1 to s_2 is determined from, in which case

$$U_{1-2} = F_c \cos\theta \int_{s_1}^{s_2} ds$$

$$U_{1-2} = F_c \cos\theta (s_2 - s_1)$$

Here the work of $\boldsymbol{F_c}$ represents the *area of the rectangle* as in Figure below:



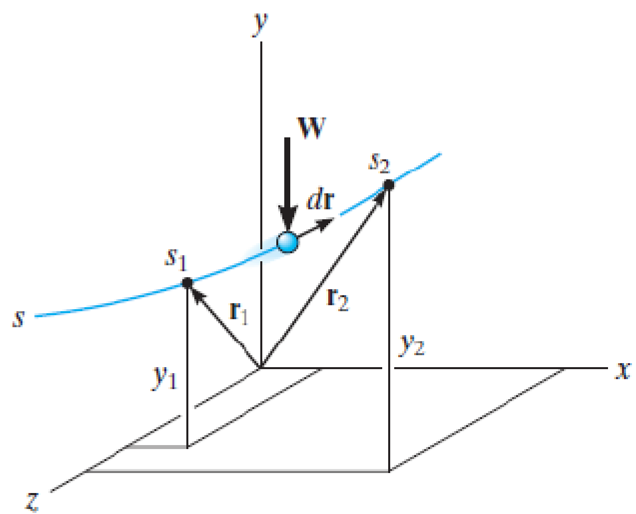
2-9-3 ***Work of a Weight***

Consider a particle of weight ***W***, which moves up along the path *s* shown in Fig. from position s_1 to position s_2 .

$$U_{1-2} = \int \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} (-W\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_{y_1}^{y_2} -W\, dy = -W(y_2 - y_1)$$

$$U_{1-2} = -W\, \Delta y$$

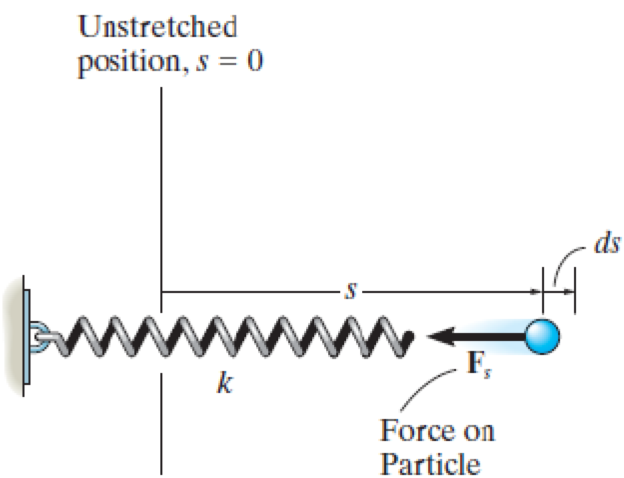


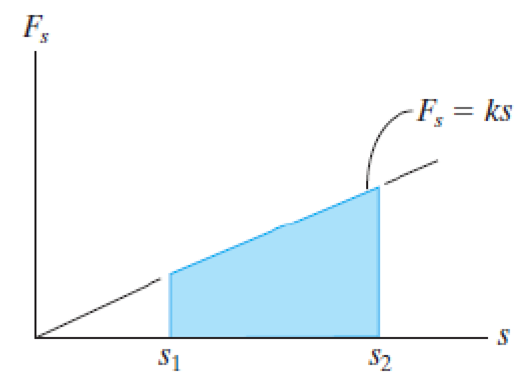
2-9-4 *Work of a Spring Force*

If an elastic spring is elongated a distance ds, then the work done by the force that acts on the attached particle is $dU = -F_s ds = -ks\ ds$. The work is *negative* since $\mathbf{F}_s$ acts in the opposite sense to ds. If the particle displaces from s_1 to s_2, the work of $\mathbf{F}_s$ is then

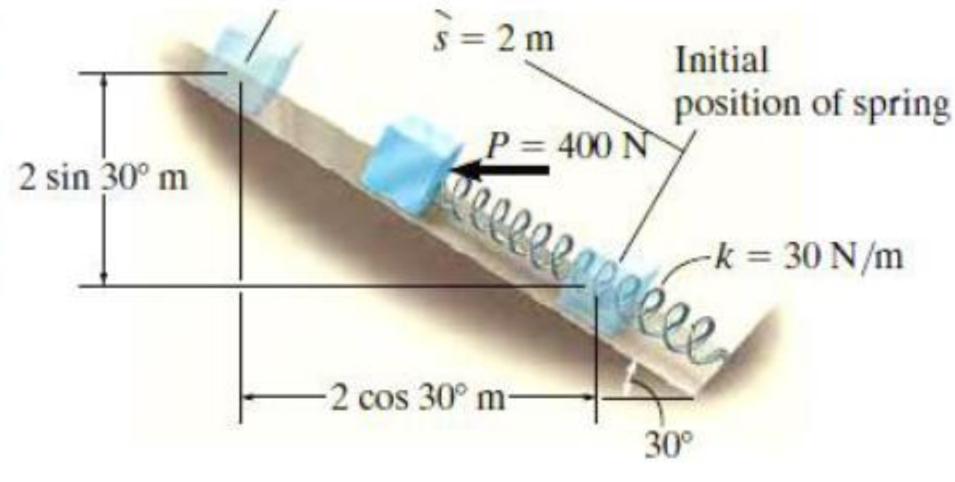
$$U_{1-2} = \int_{s_1}^{s_2} F_s\, ds = \int_{s_1}^{s_2} -ks\, ds$$

$$U_{1-2} = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$



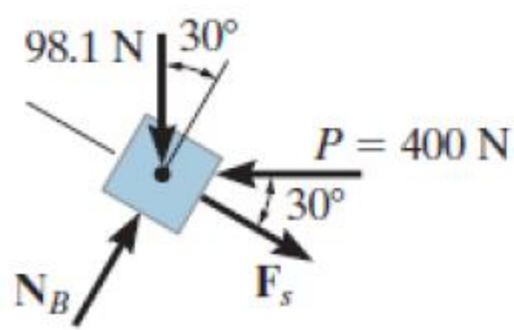


Example 9: The 10-kg block shown in Fig. rests on the smooth incline. If the spring is originally stretched 0.5 m, determine the total work done by all the forces acting on the block when a horizontal force P = 400 N pushes the block up the plane s = 2 m.



SOLUTION:

$$U_P = 400\text{ N }(2\text{ m}\cos 30^\circ) = 692.8\text{ J}$$



Spring Force F_s. In the initial position the spring is stretched $s_1 = 0.5$ m and in the final position it is stretched $s_2 = 0.5\text{ m} + 2\text{ m} = 2.5$ m. We require the work to be negative since the force and displacement are opposite to each other. The work of $\mathbf{F}_s$ is thus

$$U_s = -\left[\tfrac{1}{2}(30\text{ N/m})(2.5\text{ m})^2 - \tfrac{1}{2}(30\text{ N/m})(0.5\text{ m})^2\right] = -90\text{ J}$$

Weight W. Since the weight acts in the opposite sense to its vertical displacement, the work is negative; i.e.,

$$U_W = -(98.1\text{ N})\,(2\text{ m}\sin 30^\circ) = -98.1\text{ J}$$

Normal Force N_B. This force does *no work* since it is *always* perpendicular to the displacement.

Total Work. The work of all the forces when the block is displaced 2 m is therefore

$$U_T = 692.8\text{ J} - 90\text{ J} - 98.1\text{ J} = 505\text{ J} \qquad \textit{Ans.}$$

2-10 The Principle of Work and Energy

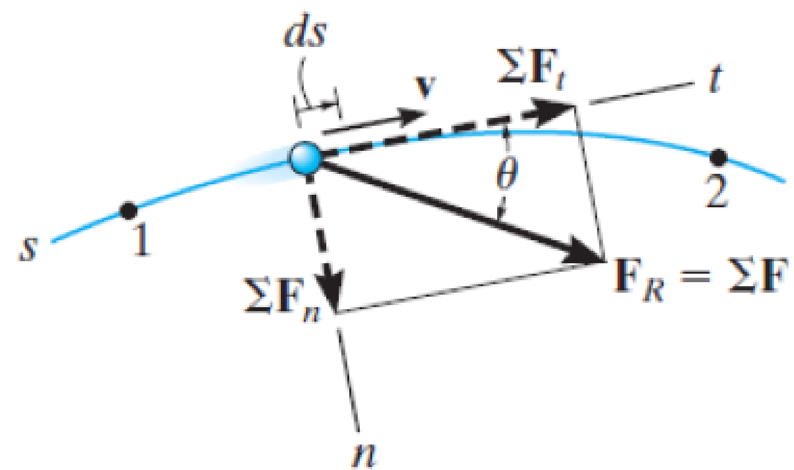
Consider the particle in Fig. which is located on the path defined relative to an inertial coordinate system. If the particle has a mass m and is subjected to a system of external forces represented by the resultant $\mathbf{F}_R = \sum \mathbf{F}$, then the equation of motion for the particle in the tangential direction is $\sum F_t = ma_t$. Applying the kinematic equation $a_t = v\, dv>ds$ and integrating both sides, assuming initially that the particle has a position $s = s_1$ and a speed $v = v_1$, and later at $s = s_2$, $v = v_2$, we have

$$\Sigma \int_{s_1}^{s_2} F_t\, ds = \int_{v_1}^{v_2} mv\, dv$$

$$\Sigma \int_{s_1}^{s_2} F_t\, ds = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

$$\Sigma U_{1-2} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

$$T_1 + \Sigma U_{1-2} = T_2$$



Where:

$\boldsymbol{T = kinetic\ energy\ \frac{1}{2}mv^2}$

Principle of Work and Energy.

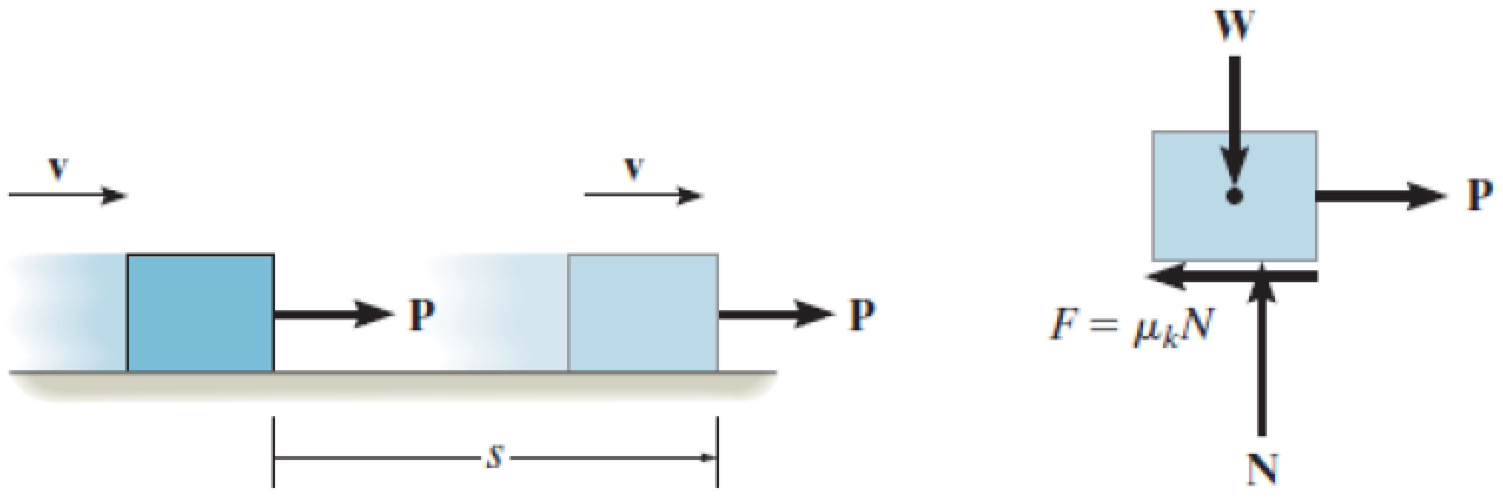
- Apply the principle of work and energy, $T_1 + \Sigma U_{1-2} = T_2$.
- The kinetic energy at the initial and final points is *always positive*, since it involves the speed squared $\left(T = \frac{1}{2}mv^2\right)$.
- A force does work when it moves through a displacement in the direction of the force.
- Work is *positive* when the force component is in the *same sense of direction* as its displacement, otherwise it is negative.
- Forces that are functions of displacement must be integrated to obtain the work. Graphically, the work is equal to the area under the force-displacement curve.
- The work of a weight is the product of the weight magnitude and the vertical displacement, $U_W = \pm Wy$. It is positive when the weight moves downwards.
- The work of a spring is of the form $U_s = \frac{1}{2}ks^2$, where k is the spring stiffness and s is the stretch or compression of the spring.

If we apply the principle of work and energy to this and each of the other particles in the system, then since work and energy are scalar quantities, the equations can be summed algebraically, which gives

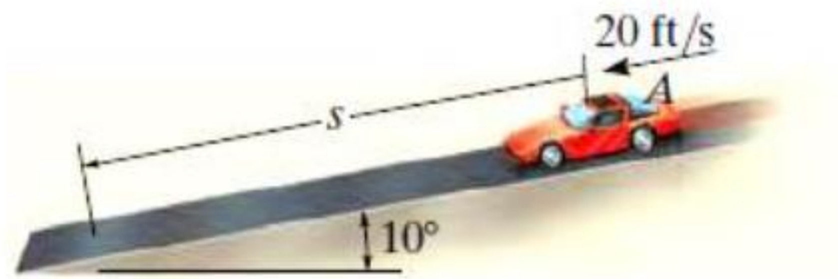
$$\Sigma T_1 + \Sigma U_{1-2} = \Sigma T_2$$

Problems involve cases where a body slides over the surface of another body in the presence of friction considers as special class of problems which requires a careful application. Consider, for example, a block which is translating a distance s over a rough surface as shown in Fig. If the applied force ***P*** just balances the *resultant* frictional force $\mu_k N$.

$$\tfrac{1}{2}mv^2 + Ps - \mu_k Ns = \tfrac{1}{2}mv^2$$



Example 10: The 3500-lb automobile shown in Fig. travels down the 10° inclined road at a speed of 20 ft/s. If the driver jams on the brakes, causing his wheels to lock, determine how far s the tires skid on the road. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.5$.



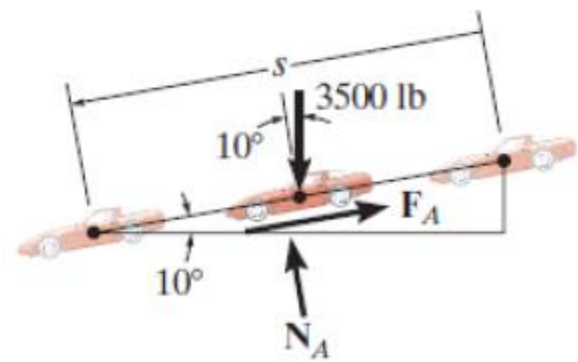
SOLUTION:

Applying the equation of equilibrium normal to the road, we have

$+\nwarrow \Sigma F_n = 0; \qquad N_A - 3500 \cos 10^\circ \text{ lb} = 0 \qquad N_A = 3446.8 \text{ lb}$

Thus,

$$F_A = \mu_k N_A = 0.5\,(3446.8 \text{ lb}) = 1723.4 \text{ lb}$$



Principle of Work and Energy.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\frac{1}{2}\left(\frac{3500 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(20 \text{ ft/s})^2 + 3500 \text{ lb}(s \sin 10^\circ) - (1723.4 \text{ lb})s = 0$$

Solving for s yields

$$s = 19.5 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If this problem is solved by using the equation of motion, *two steps* are involved. First, from the free-body diagram, Fig. , the equation of motion is applied along the incline. This yields

$$+\swarrow \Sigma F_s = ma_s; \qquad 3500 \sin 10° \text{ lb} - 1723.4 \text{ lb} = \frac{3500 \text{ lb}}{32.2 \text{ ft/s}^2} a$$

$$a = -10.3 \text{ ft/s}^2$$

Then, since a is constant, we have

$$(+\swarrow) \quad v^2 = v_0^2 + 2a_c(s - s_0);$$

$$(0)^2 = (20 \text{ ft/s})^2 + 2(-10.3 \text{ ft/s}^2)(s - 0)$$

$$s = 19.5 \text{ ft} \qquad \textit{Ans.}$$

2-11 Power and Efficiency

The term "power" provides a useful basis for choosing the type of motor or machine which is required to do a certain amount of work in a given time. For example, two pumps may each be able to empty a reservoir if given enough time; however, the pump having the larger power will complete the job sooner. The *power* generated by a machine or engine that performs an amount of work dU within the time interval dt is therefore

$$P = \frac{dU}{dt}$$

$$P = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$$

The basic units of power used in the SI and FPS systems are the watt (W) and horsepower (hp), respectively. These units are defined as

1 W = 1 J/s = 1 N . m/s

1 hp = 550 ft . lb/s

For conversion between the two systems of units, 1 hp = 746 W.

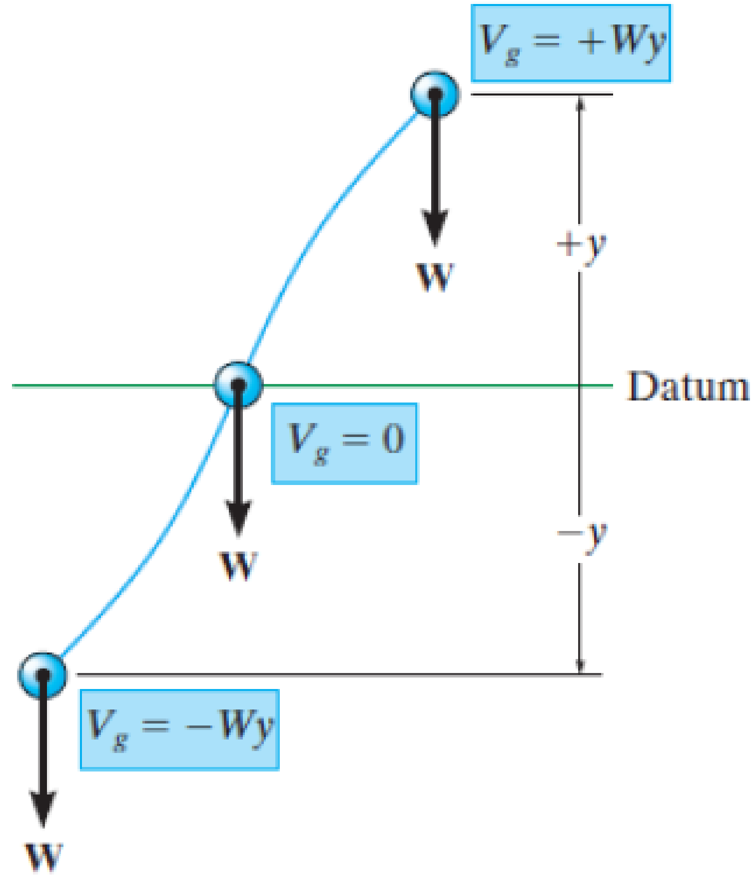
The *mechanical efficiency* of a machine is defined as the ratio of the output of useful power produced by the machine to the input of power supplied to the machine. Hence,

$$\varepsilon = \frac{\text{power output}}{\text{power input}}$$

If energy supplied to the machine occurs during the *same time interval* at which it is drawn, then the efficiency may also be expressed in terms of the ratio. Since machines consist of a series of moving parts, frictional forces will always be developed within the machine, and as a result, extra energy or power is needed to overcome these forces. Consequently, power output will be less than power input and so *the efficiency of a machine is always less than 1*. The procedure for analysis is as follow:

2-12 Conservative Forces and Potential Energy

If the work of a force is *independent of the path* and depends only on the force's initial and final positions on the path, then we can classify this force as a *conservative force*. Examples of conservative forces are the weight of a particle and the force developed by a spring. The work done by the weight depends *only* on the *vertical displacement* of the weight, and the work done by a spring force depends *only* on the spring's *elongation* or *compression*.



Energy is defined as the capacity for doing work. For example, if a particle is originally at rest, then the principle of work and energy states that $\sum U_{1\text{-}2} = T_2$. In other words, the kinetic energy is equal to the work that must be done on the particle to bring it from a state of rest to a speed v. Thus, the *kinetic energy* is a measure of the particle's *capacity to do work*, which is associated with the *motion* of the particle. When energy comes from the *position* of the particle, measured from a fixed datum or reference plane, it is called potential energy. Thus, *potential energy* is a measure of the amount of work a conservative force will do when it moves from a given position to the datum. In mechanics, the potential energy created by gravity (weight) and an elastic spring is important.

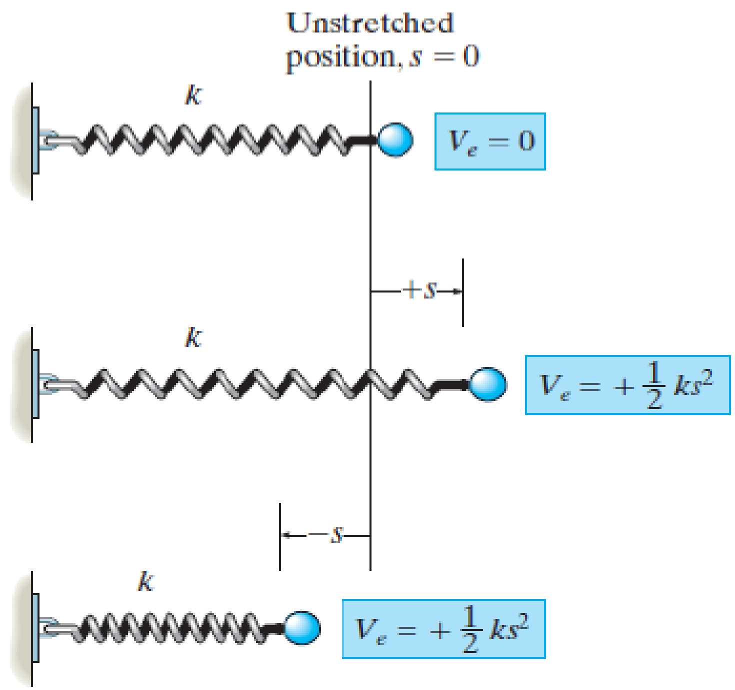
In general, if y is *positive upward*, the gravitational potential energy of the particle of weight W is

$$V_g = Wy$$

When an elastic spring is elongated or compressed a distance s from its unstretched position, elastic potential energy V_e can be stored in the spring. This energy is

$$V_e = +\frac{1}{2}ks^2$$

Here V_e is *always positive* since, in the deformed position, the force of the spring has the *capacity* or "potential" for always doing positive work on the particle when the spring is returned to its unstretched position.



Elastic potential energy

In the general case, if a particle is subjected to both gravitational and elastic forces, the particle's potential energy can be expressed as a *potential function*, which is the algebraic sum

$$V = V_g + V_e$$

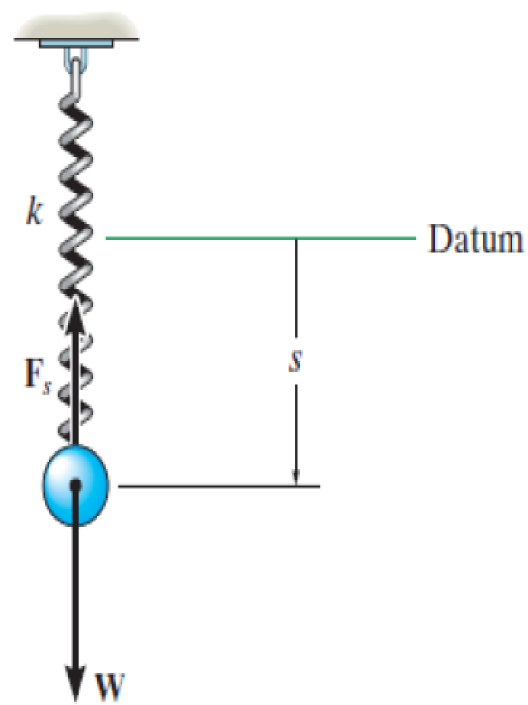
The work done by a conservative force in moving the particle from one point to another point is measured by the *difference* of this function, i.e.,

$$U_{1-2} = V_1 - V_2$$

For example, the potential function for a particle of weight W suspended from a spring can be expressed in terms of its position, s, measured from a datum located at the unstretched length of the spring, We have

$$V = V_g + V_e$$

$$= -Ws + \tfrac{1}{2}ks^2$$



If the particle moves from $s1$ to a lower position s_2, it can be seen that the work of **W** and $\mathbf{F}_s$ is

$$U_{1-2} = V_1 - V_2 = \left(-Ws_1 + \tfrac{1}{2}ks_1^2\right) - \left(-Ws_2 + \tfrac{1}{2}ks_2^2\right)$$

$$= W(s_2 - s_1) - \left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$

2-13 Conservation of Energy

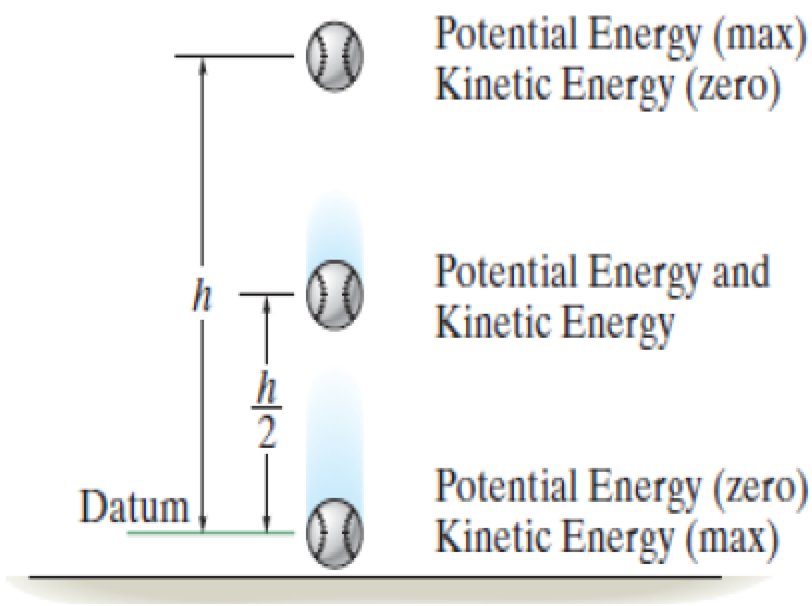
When a particle is acted upon by a system of *both* conservative and nonconservative forces, the portion of the work done by the *conservative forces* can be written in terms of the difference in their potential energies , i.e., $(\sum U_{1\text{-}2})_{\text{cons.}} = V_1 - V_2$. As a result, the principle of work and energy can be written as

$$T_1 + V_1 + (\Sigma U_{1-2})_{\text{noncons.}} = T_2 + V_2$$

Here $(\sum U_{1\text{-}2})_{\text{noncons.}}$ represents the work of the nonconservative forces acting on the particle. If *only conservative forces* do work then we have

$$T_1 + V_1 = T_2 + V_2$$

This equation is referred to as the *conservation of mechanical energy* or simply the *conservation of energy*. It states that during the motion the sum of the particle's kinetic and potential energies remains *constant*. For this to occur, kinetic energy must be transformed into potential energy, and vice versa. For example, if a ball of weight **W** is dropped from a height h above the ground (datum), the potential energy of the ball is maximum before it is dropped, at which time its kinetic energy is zero. The total mechanical energy of the ball in its initial position is thus



$$E = T_1 + V_1 = 0 + Wh = Wh$$

When the ball has fallen a distance $h>2$, its speed can be determined by using

$$v^2 = v_0^2 + 2a_c(y - y_0)$$

which yields

$$v = \sqrt{2g(h/2)} = \sqrt{gh}.$$

The energy of the ball at the mid-height position is therefore

$$E = T_2 + V_2 = \frac{1}{2}\frac{W}{g}(\sqrt{gh})^2 + W\left(\frac{h}{2}\right) = Wh$$

Just before the ball strikes the ground, its potential energy is zero and its speed is

$$v = \sqrt{2gh}$$

Here, again, the total energy of the ball is

$$E = T_3 + V_3 = \frac{1}{2}\frac{W}{g}(\sqrt{2gh})^2 + 0 = Wh$$

Note that when the ball comes in contact with the ground, it deforms somewhat, and provided the ground is hard enough, the ball will rebound off the surface, reaching a new height h', which will be *less* than the height h from which it was first released. Neglecting air friction, the difference in height accounts for an energy loss,

$$El = W(h - h')$$

Which occurs during the collision. Portions of this loss produce noise, localized deformation of the ball and ground, and heat.

If a system of particles is *subjected only to conservative forces*, then an equation can be written for the particles. Applying the ideas of the preceding discussion, ($\sum T_1 + \sum U_{1\text{-}2} = \sum T_2$) becomes

$$\Sigma T_1 + \Sigma V_1 = \Sigma T_2 + \Sigma V_2$$

Here, the sum of the system's initial kinetic and potential energies is equal to the sum of the system's final kinetic and potential energies. In other words, $\sum T + \sum V = \text{const.}$ The conservation of energy equation can be used to solve problems involving *velocity*, *displacement*, and *conservative force systems*. It is generally *easier to apply* than the principle of work and energy because this equation requires specifying the particle's kinetic and potential energies at only *two points* along the path, rather than determining the work when the particle moves through a *displacement*. For application it is suggested that the following procedure be used.

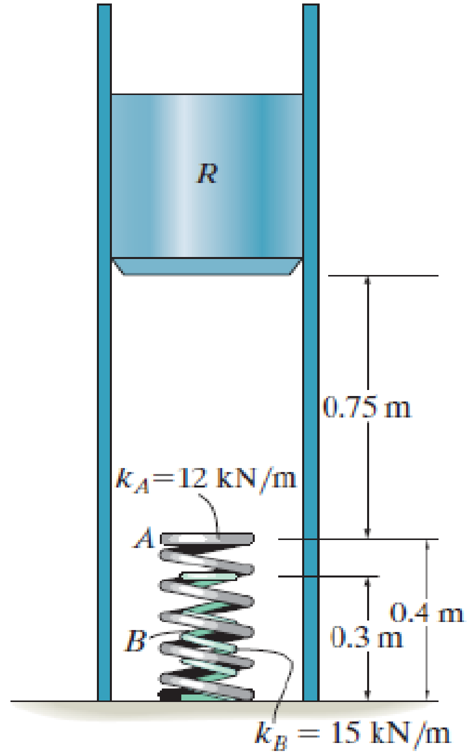
Potential Energy.

- Draw two diagrams showing the particle located at its initial and final points along the path.
- If the particle is subjected to a vertical displacement, establish the fixed horizontal datum from which to measure the particle's gravitational potential energy Vg .
- Data pertaining to the elevation y of the particle from the datum and the stretch or compression s of any connecting springs can be determined from the geometry associated with the two diagrams.
- Recall $Vg = Wy$, where y is positive upward from the datum and negative downward from the datum; also for a spring, Ve = 12 ks2, which is *always positive*.

Conservation of Energy.

- Apply the equation $T1 + V1 = T2 + V2$.
- When determining the kinetic energy, T = 12 mv2, remember that the particle's speed v must be measured from an inertial reference frame.

Example 13: The ram R shown in Fig. has a mass of 100 kg and is released from rest 0.75 m from the top of a spring, A, that has a stiffness k_A = 12 kN/m. If a second spring B, having a stiffness k_B = 15 kN/m, is "nested" in A, determine the maximum displacement of A needed to stop the downward motion of the ram. The unstretched length of each spring is indicated in the figure. Neglect the mass of the springs.



SOLUTION:

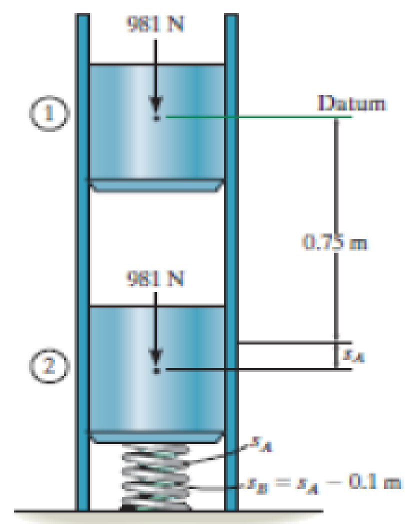
Potential Energy. We will *assume* that the ram compresses *both* springs at the instant it comes to rest. The datum is located through the center of gravity of the ram at its initial position, Fig. When the kinetic energy is reduced to zero ($v_2 = 0$), A is compressed a distance s_A and s_B compresses $s_B = s_A$ - 0.1 m.

Conservation of Energy.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = 0 + \left\{\tfrac{1}{2}k_A s_A^2 + \tfrac{1}{2}k_B(s_A - 0.1)^2 - Wh\right\}$$

$$0 + 0 = 0 + \left\{\tfrac{1}{2}(12\,000\ \text{N/m})s_A^2 + \tfrac{1}{2}(15\,000\ \text{N/m})(s_A - 0.1\ \text{m})^2 - 981\ \text{N}\,(0.75\ \text{m} + s_A)\right\}$$



Rearranging the terms,

$$13\,500 s_A^2 - 2481 s_A - 660.75 = 0$$

Using the quadratic formula and solving for the positive root, we have

$$s_A = 0.331 \text{ m} \qquad \textit{Ans.}$$

Since $s_B = 0.331 \text{ m} - 0.1 \text{ m} = 0.231 \text{ m}$, which is positive, the assumption that *both* springs are compressed by the ram is correct.

NOTE: The second root, $s_A = -0.148 \text{ m}$, does not represent the physical situation. Since positive s is measured downward, the negative sign indicates that spring A would have to be "extended" by an amount of 0.148 m to stop the ram.