University of Anbar

Collage of Science

Department of Applied Mathematics

Third Year – First Semester

Lectures in Mathematical analysis

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Lecture No. 1

Basic definitions

Mathematical analysis 1

Third Class

- Real Numbers R and Extended Real Numbers R*
- Euclidian space o Countable and uncountable sets.
- Metric S paces
    - Compactness
    - Connectedness.
    - Perfect sets
- Convergence and divergence in Metric Space
- Cauchy sequence
- Absolute and conditional convergence .
- Product of series
- Compactness and connectedness .
- Continuity and uniform continuity
- Intermediate value theorem
- Sequence and series of functions .
- uniform and point wise continuity
- power series

Basic definitions

Definition 1. (Natural numbers) $\mathbb{N}$

The set { 1,2,3, … } is called Naturl numbers denoted by $\mathbb{N}$

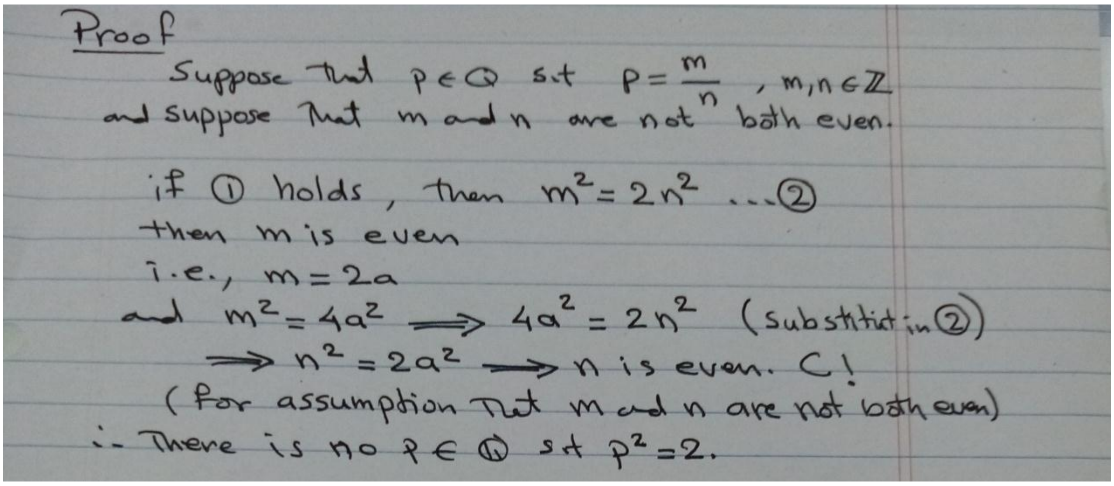
Definition 2. (integer numbers $\mathbb{Z}$)The set $\{\dots, -2, -1, 0,1,2, \dots\}$ is called integer numbers and denoted by $\mathbb{Z}$.

Definition 3. (Rational numbers $\mathbb{Q}$)

The set $\mathbb{Q} = \{\frac{p}{q}, p, q \in \mathbb{Z}\ and\ q \neq 0\}$ is called the rational numbers and denoted by $\mathbb{Q}$

The set Q is not sufficient for all purposes, for example, there is no rational number $p \in \mathbb{Q}$ s.t $p^2 = 2$. This problem leads to define the set of irrational numbers The set of irrational numbers is denoted by $\mathbb{Q}'$ and the set of all real numbers is the union of $\mathbb{Q}$ and $\mathbb{Q}'$

Theorem 1. Show that The equation $p^2 = 2$. has no solution in $\mathbb{Q}$


Proof

Suppose that $p \in \mathbb{Q}$ s.t $p = \frac{m}{n}$, $m, n \in \mathbb{Z}$
and suppose that m and n are not both even.

if ① holds, then $m^2 = 2n^2$ ...②
then m is even
i.e., $m = 2a$
and $m^2 = 4a^2 \Longrightarrow 4a^2 = 2n^2$ (substitut in ②)
$\Longrightarrow n^2 = 2a^2 \Longrightarrow n$ is even. C!
(for assumption that m and n are not both even)
$\therefore$ There is no $p \in \mathbb{Q}$ s.t $p^2 = 2$.

To explain the relation between $\mathbb{Q}$ and $\mathbb{Q}'$ we need the following theorem:

Theorem2: If $\frac{m}{n}$ and $\frac{p}{q} \in \mathbb{Q}$, $p \neq 0$ then

$(\frac{m}{n} + \sqrt{2}\frac{p}{q})$ is irrational.

Proof:

Suppose that $(\frac{m}{n} + \sqrt{2}\frac{p}{q})$ is rational number

$\longrightarrow \exists\, r, s \in \mathbb{Z}$, $s \neq 0$

such that $\frac{m}{n} + \sqrt{2}\frac{p}{q} = \frac{r}{s} \longrightarrow \sqrt{2} = \frac{q}{p}(\frac{r}{s} - \frac{m}{n})$

$= \frac{q(rn - sm)}{psn}$, $psn \neq 0 \longrightarrow$ C!

Since $\frac{q(rn-sm)}{psn} \in \mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$ (by Th1)

$\longrightarrow \frac{m}{n} + \sqrt{2}\frac{p}{q}$ is irrational numbers.

Theorem 3: *Between any two distanct rationals there isnan irrational number.*

Proof:

Let $\frac{m}{n}$, $\frac{p}{q}$ be two rationals and

$$\frac{m}{n} < \frac{p}{q} \quad \rightarrow \frac{p}{q} - \frac{m}{n} > 0$$

$$\frac{m}{n} < \frac{m}{n} + \frac{\sqrt{2}}{2}\,(\frac{p}{q} - \frac{m}{n})$$

Since $\frac{\sqrt{2}}{2} < 1 \ \rightarrow \frac{m}{n} < \frac{m}{n} + \frac{\sqrt{2}}{2}\,(\frac{p}{q} - \frac{m}{n})$

$\rightarrow \qquad\qquad < \frac{m}{n} + 1\,\left(\frac{p}{q} - \frac{m}{n}\right) = \frac{p}{q}$

Since (by theorem 2) $\ \frac{m}{n} + \frac{\sqrt{2}}{2}\,(\frac{p}{q} - \frac{m}{n})$ is irrational .

Then there is an irrational between $\frac{m}{n}$ and $\frac{p}{q}$

Definition :

A decimal is an expression of the form :

$\pm a_0.a_1a_2a_3 \ldots$ such that a_0 is non-negative and $a_1, a_2, a_3, \ldots$ are called digits which may:

- **Finite decimal**
- **Infinite decimal** which may be ***recurring*** or ***non-recurring***

Example:

1- $2.3456 = 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{6}{10000} = \frac{23456}{10^4}$

2- $\frac{3103}{9990} = 0.3016016016 \ldots = 0.3\overline{016}$ infinite recurring.

Theorem 4: *Between any two distanct irrationals there isn a rational number.*

Proof:

Let a and b be two irrational and a<b. since $a \neq b$ then there is a first decimal digit where a and b differ.

$$\rightarrow a = a_0.a_1a_2a_3 \ldots a_{n-1}a_n$$

$$\rightarrow b = b_0.b_1b_2b_3 \ldots b_{n-1}b_n$$

and $a_n < b_n$

chose $x = a_0.a_1a_2a_3 \ldots a_{n-1}b_n$ then $a < x < b$

and $= a_0.a_1a_2a_3 \ldots a_{n-1}b_n = \frac{=a_0a_1a_2a_3 \ldots a_{n-1}b_n}{10^n} = a$

example

let $\left.\begin{matrix} a=2.12345153 \\ b=2.12345342 \end{matrix}\right\} a < b$

$$x = 2.12345300, \quad a < x < b$$

Exercise : Give another proof for theorem 3 and theorem 4.

References

1- Principles Of Mathematical Analysis - W.Rudin.
https://59clc.files.wordpress.com/2012/08/functional-analysis-_-rudin-2th.pdf