University of Anbar

Collage of Science

Department of Applied Mathematics

**Third Year – First Semester**

# Lectures in Mathematical analysis

By: Dr. Rifaat Saad Abdul-Jabbar

## Lecture No. 2

## Basic definitions

Definition

We say that the set $F$ is field if there exist two binary operations $+\ and\ \cdot$ such that:

$+: F \times F \to F$

$(x, y) \to x + y \in F$

$\cdot : F \times F \to F$

$(x, y) \to x \cdot y \in F, \quad x, y \neq 0$

and :

1. $(F, +)$ is commutative group
2. $(F \backslash \{0\}, \cdot)$ is commutative group.
3. $\left.\begin{array}{l} x \cdot (y + z) = x \cdot y + x \cdot z \\ (x + y) \cdot z = x \cdot z + y \cdot z \end{array}\right\} \ \forall\, x, y, z \in F$ distribution law

We write $(F, +, \cdot)$ is a field.

Definition (ordered field)

We say that the field F is ordered if:

1. $\forall\, x, y \in F$, one and only one relation is true:

   $x < y, \qquad y < x \qquad or\ x = y$
2. If $a < b$ and $b < c$ then $a < c$
3. If $a < b$ then $a + c < b + c$ , $\forall c \in F$. $+$ preserves ordering.
4. $0 < x$ and $0 < y$ then $0 < xy$
5. $x < y$ and $0 < z$ then $xz < yz$

   Then $(F, +, \cdot, <)$ is ordered field.

The axiom of arithmetic of $\mathbb{R}$ (field)

Define two functions:

$+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ (x, y) \to x + y \in \mathbb{R}$

$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ (x, y) \to x \cdot y \in \mathbb{R}$

And

A1- $(a + b) + c = a + (b + c)$

A2- $a + b = b + a$

A3- there exists a unique element $0 \in \mathbb{R} \ni a + 0 = 0 + a = a$

A4- $\forall a \in \mathbb{R}$, there exists a unique element $x \in \mathbb{R} \ni a + x = a + x = 0$ .

*(A1- A4) imply that $(\mathbb{R}, +)$ is commutative group.*

A5- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

A6- $a \cdot b = b \cdot a$

A7- there exists a unique element $1 \in \mathbb{R} \ni a \cdot 1 = 1 \cdot a = a$

A8- $\forall a \neq 0, a \in \mathbb{R}$, there exists a unique element $x \in \mathbb{R}$

$\ni a \cdot x = a \cdot x = 1$ .

i.e. $x = a^{-1} = \frac{1}{a}$

*(A5- A8) imply that* $(\mathbb{R}\backslash\{0\},\cdot)$ *is commutative group.*

A9- distribution law : $a \cdot (b + c) = a \cdot b + a \cdot c$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

*(A1- A9) imply that* $(\mathbb{R}, +,\cdot)$ *is a field.*

A10- either $a < b, b < a$ or $a = b$

A11- $a < b$ and $b < c \to a < c$

A12- $0 < a$ and $0 < b \to 0 < a \cdot b$

A13- $a < b \to a + c < b + c, \quad \forall c \in \mathbb{R}$.

A14-$a < b$ and $a < c$ then $a \cdot c < b \cdot c$

*(A1- A14) imply that* $(\mathbb{R}, +, \cdot, <)$ *is an ordered field.*

$$\mathbb{R}^{+} = \{x \in \mathbb{R}, 0 < x \text{ or } x = 0\}$$

$$\mathbb{R}^{-} = \{x \in \mathbb{R}, x < 0 \text{ or } x = 0\}$$

$$\mathbb{R}^{++} = \{x \in \mathbb{R}, 0 < x \}$$

Let $a \in \mathbb{R}$, $a$ is called : $\begin{cases} positive \text{ if } 0 < a \\ negative \text{ if } a < 0 \\ non-negative \text{ if } 0 < a \text{ or } a = 0 \\ non-positive \text{ if } a < 0 \text{ or } a = 0 \end{cases}$

Ex: $\mathbb{Q}$ is an ordered field

**Exercise**: prove that $0 \leq x^2$ for any $x \in \mathbb{R}$

Definition: Absolute value

Let $a \in \mathbb{R}$ the absolute value of a is defined as:

$$|a| = \begin{cases} a & \text{if } 0 > a \\ 0 & \text{if } a = 0 \\ -a & \text{if } a > 0 \end{cases}$$

On the other hand, we can see $|.|$ is a function defined on $\mathbb{R}$ into $\mathbb{R}^{+}$ i.e. $|.| : \mathbb{R} \to \mathbb{R}^{+}$.

Properties of absolute value function:

1. $a \leq$
2. $|a|^2 \leq a^2$
3. $|ab| = |a||b|$ and $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$
4. $|a + b| \leq |a| + |b|$

$|a-b| \geq \big||a|-|b|\big|$

5. $|a| = \max\{a,-a\}, -|a| = \min\{a,-a\}$
6. If $\epsilon > 0$ then $|a-b| < \epsilon \leftrightarrow b-\epsilon < a < b+\epsilon$

Definition : let $S \subset \mathbb{R}$, $S$is said to be bounded above if there is same real M such that: $x \leq M \quad \forall x \in S$.

M is called upper bound for S.

Definition $M'$ is called the least upper bound (or supremum) of S if:

1- $M'$ is an upper bound of S
2- $M' \leq M$ for all M, where M is an upper bound of S.

Notes: 1. $M'$ (if exists) is unique (check)

2. we denote to $M'$ by L.U.B(S) or $sup(S)$

**Definition** Let $S \subset \mathbb{R}$, $S$ is said to be bounded below if there is some $m \in \mathbb{R}$ s. t. $m \leq x, \forall x \in S$

$m$ is called lower bound

**Definition** $m'$ is called the greatest lower bound (or infemum) of S if:

1- $m'$ is a lower bound of S
2- $m \leq m'$for all $m$, where $m$ is an lower bound of $S$.

Notes: 1. $m'$ (if exists) is unique (check)

2. we denote to $m'$ by glb(S) or $inf(S)$

Definition : Let $S \subset \mathbb{R}$, S is called:

1. Unbounded above if there is no upper bound.
2. Unbounded below if there is no lower bound.
3. Unbounded= Unbounded above+ Unbounded below

Exercises

1. Prove that $\mathbb{R}^+ = \{x \in \mathbb{R}, x \geq 0\}$ is bounded below and unbounded above.
2. Prove that $\mathbb{R}^- = \{x \in \mathbb{R}, x \leq 0\}$ is bounded above and unbounded below.
3. Find $\sup(S)$ and $\inf(S)$ where:
   $S = [0,1)$, $S = [0,1)$, $S = \mathbb{N}$
   $S = (0,\infty)$, $S = \{\frac{1}{n},\ n = 1,2,3,\dots\}$
   $S = \{x \in \mathbb{R}, x^2 - 9 > 0\}$
4. Let $A, B$ be two bounded sets and $A \subseteq B$.
   Prove that: $a)$ $\sup(A) \leq \sup(B)$
   $b)$ $\inf(B) \leq \inf(A)$
5. If $a$, $b$ are two real numbers $\ni a \leq b + \epsilon$, $\forall \epsilon > 0$, then $a \leq b$

## References

1- Principles Of Mathematical Analysis - W.Rudin.
https://59clc.files.wordpress.com/2012/08/functional-analysis-_-rudin-2th.pdf