University of Anbar

Collage of Science

Department of Applied Mathematics

Third Year – First Semester

Lectures in Mathematical analysis

By: Dr. Rifaat Saad Abdul-Jabbar

Lecture No. 3

Archimedean field

Completing axioms of $\mathbb{R}$

A15- Every nonempty set of real numbers that is bounded above has a supremum.

OR:

Every nonempty set of real numbers that is bounded below has a infimum.

(A1- A15) imply that $\mathbb{R}$ is complete ordered field.

Definition: Isomorphism of ordered field

Let F and K be two ordered fields, we say that $\phi: F \to K$ is isomorphism if ϕ is one to one , onto and preserve $(+,\cdot,<)$

i.e. for all $x, y \ in\ F$, $\phi(x+y) = \phi(x) + \phi(y)$

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y)$$

$$if\ x < y \text{ then } \phi(x) < \phi(y)$$

Theorem: (Uniqueness of complete ordered field)-without proof

Every two complete ordered fields are isomorphic.

Extended real number

It is convenient to extend the system of real numbers by adding two elements ∞ and$-\infty$ so $\mathbb{R}$ will be call extended real numbers, and:

If $x \in \mathbb{R} \to \ -\infty < x < \infty$

$$x + \infty = \infty + x = \infty = -x + \infty$$

$$x - \infty = -\infty + x = -\infty - x = -\infty$$

$$x \cdot \infty = \infty \cdot x = \infty \ if\ x > 0$$

$$x \cdot \infty = \infty \cdot x = -\infty \ if\ x < 0$$

Note that the following combinations are undefined:

$$\infty - \infty \ , -\infty + \infty, \quad 0 \cdot \infty \quad , \ \infty \cdot 0$$

Exercises

6. if $\phi \neq S \subset \mathbb{R}$, $u = \sup(S)$ then every $p < u, \exists x \in S \ni p < x \le u$.
   i.e., if $u = \sup(S)$, then $\forall \epsilon > 0$, $\exists x \in S \ni u - \epsilon < x \le u$
7. If $v = inf(S)$ then $\forall p,\ p > v,\ \exists x \in S \ni \ v \le x < p$.
8. If $A,\ B$ are two bounded subsets of $\mathbb{R}$, $t \in \mathbb{R}$.

   Define $A + B = C = \{x + y, x \in A, y \in B\}$

   $tA = \{t \cdot x\ x \in A\}$

Then:

$$\sup(A + B) = \sup(A) + \sup(B)$$

$$\inf(A + B) = \inf(A) + \inf(B)$$

$$\sup(tA) = \begin{cases} t\cdot\sup(A) & if\ t>0 \\ t\cdot\inf(A) & if\ t<0 \end{cases}$$

$$\inf(tA) = \begin{cases} t\cdot\inf(A) & if\ t>0 \\ t\cdot\sup(A) & if\ t<0 \end{cases}$$

Theorem 5

The set $\mathbb{N}$ is not bounded above.

Proof

Suppose that $\mathbb{N}$ is bounded above then $\mathbb{N}$ has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$

By example 6, $\exists n \in \mathbb{N} \ni u - 1 < n$

$\rightarrow u < n + 1$ for this n.

Since $n + 1 \in \mathbb{N} \rightarrow C!$

(for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded)

Theorem 6

For every real number x $\exists n \in \mathbb{N},\ n > x$.

Proof

Suppose not, $n \leq x, \forall\, n \in \mathbb{N}$

Then x is upper bound of $\mathbb{N}$ $\rightarrow C!$ (theorem 5)

Then there must exists $n \in \mathbb{N} \ni n > x$.

Theorem 7 Archimedean property

If $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R},\ \exists n \in \mathbb{N} \ni nx < y$

Proof

If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \geq y$, $\forall n \in \mathbb{N}$

$\rightarrow n \leq \frac{y}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C!$ (by Th. 5)

Definition

Let F be any field , F is called Archimedean field if $\mathbb{N}$ is unbounded in F.

i.e., $\forall x \in F,\ \exists n \in \mathbb{N} \ni n > x$

Example 1- $\mathbb{R}$ is arch. Field (by Th. 6)

2- $\mathbb{Q}$ is arch. Field

Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n};\ n \in \mathbb{N}\}$

Theorem 8 (Density of real numbers)

a- Let $x, y \in \mathbb{R}$ and $x < y$, then $\exists$ infinitely rationals between x and y

b- Let $x, y \in \mathbb{R}$ and $x < y$, then $\exists$ infinitely irrationals between x and y.

Proof :

a) Since $x < y$ then $y - x > 0$

by arch. Prop. $\exists\, n \in \mathbb{N} \ni \ n(y-x) > 1$

$y - x > \frac{1}{n}$ since nx and $-nx \in \mathbb{R}$

$$\xrightarrow{by\ theorem\ 6} \exists m, m' \ni m > nx \text{ and } m' > -nx$$

$$\rightarrow\ -m' < nx < m \rightarrow \frac{m'}{n} < x < \frac{m}{n}$$

since the set $\{-m', -m'+1, \dots, m\}$ is finite

$\rightarrow$ let $m''$the smallest number such that $x \le \frac{m''}{n}$

($x \le \frac{m''}{n}$ هي اصغر عدد في المجموعة يحقق المتراجحة m'')

Since $m'' - 1 < m'' \rightarrow \frac{m''-1}{n} < x$ (حسب اختيار m'')

We have

$$x < \frac{m''}{n} = \frac{m''-1}{n} + \frac{1}{n} \le x + \frac{1}{n}$$

$< x + (y - x) = y \rightarrow r = \frac{m''}{n}$ is rational and $x < r < y$

Continue in this way to find $r_i \in \mathbb{Q}$ and $x < r_i < r$,

$$and\ x < \cdots < r_2 < r_1 < r < y$$

Remark: $\mathbb{Q}$ is ordered field and arch. Prop, but $\mathbb{Q}$ is not complete.

Theorem 9

$\mathbb{Q}$ is not complete field.

Proof

Suppose $\mathbb{Q}$ is complete.

Let $S = \{x : x \in \mathbb{Q}^+ \ and\ x^2 \le 2\} \subset [0, \sqrt{2}) \subset \mathbb{R}$

$\rightarrow S \ne \phi$ since $1 \in S$ and S is bounded above by ... $4 < 3 < 2 < \sqrt{2}$

By completeness , $\exists r \in \mathbb{Q} \ni r = \sup(S)$, axiom of $\mathbb{Q}$

Suppose $T = \{x \in \mathbb{R}^+ : x^2 \le 2\} = [0, \sqrt{2}]$

$\rightarrow S \subset T$ and $\sqrt{2}$ is irrational upper bound of S

$\rightarrow r < \sqrt{2}$

By Th.8, $\exists\, \bar{r} \in \ \mathbb{Q} \ni r < \bar{r} < \sqrt{2}$

$\rightarrow (\bar{r})^2 < 2 \rightarrow \bar{r} \in S \rightarrow C!$

Since $r = \sup(S) < r^2 \rightarrow\ \mathbb{Q}$ is not complete.

Theorem (10) (WITHOUT PROOF)

The equation $x^2 = 2$ has a unique positive solution in $\mathbb{R}$

References

1- Principles Of Mathematical Analysis - W.Rudin.
https://59clc.files.wordpress.com/2012/08/functional-analysis-_-rudin-2th.pdf