University of Anbar

Collage of Science

Department of Applied Mathematics

Third Year – First Semester

Lectures in Mathematical analysis

By: Dr. Rifaat Saad Abdul-Jabbar

Lecture No. 3

Archimedean field

University of Anbar		Third Year — First Semester
Department of Applied Mathematics		Lectures in Mathematical analysis
Collage of Science	Lecture No. 3	By: Dr. Rifaat Saad Abdul-Jabbar

Completing axioms of \mathbb{R}

A15- Every nonempty set of real numbers that is bounded above has a supremum.

OR:

Every nonempty set of real numbers that is bounded below has a infimum.

(A1- A15) imply that \mathbb{R} is complete ordered field.

Definition: Isomorphism of ordered field

Let F and K be two ordered fields, we say that $\phi: F \to K$ is isomorphism if ϕ is one to one , onto and preserve (+,,<)

i.e. for all x, y in F, $\phi(x + y) = \phi(x) + \phi(y)$ $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ if x < y then $\phi(x) < \phi(y)$

Theorem: (Uniqueness of complete ordered field)-without proof

Every two complete ordered fields are isomorphic.

Extended real number

It is convenient to extend the system of real numbers by adding two elements ∞ and $-\infty$ so \mathbb{R} will be call extended real numbers, and: If $x \in \mathbb{R} \to -\infty < x < \infty$

 $x + \infty = \infty + x = \infty = -x + \infty$ $x - \infty = -\infty + x = -\infty - x = -\infty$ $x \cdot \infty = \infty \cdot x = \infty \text{ if } x > 0$ $x \cdot \infty = \infty \cdot x = -\infty \text{ if } x < 0$

Note that the following combinations are undefined:

 $\infty - \infty$, $-\infty + \infty$, $0 \cdot \infty$, $\infty \cdot 0$

Exercises

6. if
$$\phi \neq S \subset \mathbb{R}$$
, $u = \sup(S)$ then every $p < u, \exists x \in S \ni p < x \le u$.
i.e., if $u = \sup(S)$, then $\forall \epsilon > 0$, $\exists x \in S \ni u - \epsilon < x \le u$
7. If $v = inf(S)$ then $\forall p, p > v, \exists x \in S \ni v \le x < p$.

8. If *A*, *B* are two bounded subsets of \mathbb{R} , $t \in \mathbb{R}$.

Define
$$A + B = C = \{x + y, x \in A, y \in B\}$$

 $tA = \{t \cdot x \ x \in A\}$

Then:

sup(A + B) = sup(A) + sup(B)inf(A + B) = inf(A) + inf(B)

oldge of Science Letture No.3 by Dr. Rifact Saad Abdul/Jabb sup(tA) = { t · sup(A) if t > 0 t · inf(A) if t < 0 if t < 0 inf(tA) = { t · inf(A) if t > 0 t · sup(A) if t < 0 Theorem 5 The set N is not bounded above. Proof Suppose that N is bounded above then N has a supremum (by complete axion) say u = sup(N) By example 6, ∃n ∈ N ∋ u − 1 < n → u < n + 1 for this n. Since n + 1 ∈ N → C! (for sup(N) = u < n + 1 ∈ N → N is bounded) Cheorem 6 For every real number x ∃n ∈ N, n > x. Proof Suppose not, n ≤ x, ∀ n ∈ N Cheorem 7 Archimedean property fx ∈ ℝ ⁺⁺ , then for any y ∈ ℝ, ∃n ∈ N ∋ nx < y Proof If there is no n ∈ N ∋ nx < y, we have nx ≥ y, ∀n ∈ N → n ≤ $\frac{y}{x}$, ∀n ∈ N → $\frac{y}{x}$ is an upper bound for N → C! (by Th. 5) Definition Let F be any field , F is called Archimedean field if N is inbounded in F. .e., ∀x ∈ F, ∃n ∈ N ∋ n > x Brobounded in F. .e., ∀x ∈ F, ∃n ∈ N ∋ n > x Example 1- ℝ is arch. Field (by Th. 6) .2 · Q is arch. Field Exercise: prove that sup(S) = $\frac{1}{2}$ where S = $\{\frac{n-1}{2n}$; n ∈ N} Cheorem 8 (Densit	University of Anbar Department of Applied Mathematics		Third Year — First Semester Lectures in Mathematical analysis
$\sup(tA) = \begin{cases} t \cdot \sup(A) & if t > 0 \\ t \cdot \inf(A) & if t < 0 \\ inf(tA) = \begin{cases} t \cdot \inf(A) & if t < 0 \\ t \cdot \sup(A) & if t < 0 \end{cases}$ Theorem 5 The set N is not bounded above. Proof Suppose that N is bounded above then N has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$ By example $6, \exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C!$ (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \rightarrow C!$ (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{x}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{x}{x}$ is an upper bound for $\mathbb{N} \rightarrow C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R} \text{ and } x < y$, then \exists infinitely rationals between x and y $b \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between x and y	Collage of Science	Lecture No. 3	By: Dr. Rifaat Saad Abdul-Jabbar
$\inf(tA) = \begin{cases} t \cdot \inf(A) & \text{if } t > 0 \\ t \cdot \sup(A) & \text{if } t < 0 \end{cases}$ Theorem 5 The set N is not bounded above. Proof Suppose that N is bounded above then N has a supremum (by complet axiom) say $u = \sup(\mathbb{N})$ By example $6, \exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C!$ (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Chen x is upper bound of $\mathbb{N} \rightarrow C!$ (theorem 5) Chen there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if N is ambounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example $1 - \mathbb{R}$ is arch. Field (by Th. 6) $2 - \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a - \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y $b - \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	$\sup(tA) = \begin{cases} t \cdot \sup(A) \\ t \cdot \inf(A) \end{cases}$) $if t > 0$ if t < 0	
Theorem 5 The set N is not bounded above. Proof Suppose that N is bounded above then N has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$ By example $6, \exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if N is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example $1 \cdot \mathbb{R}$ is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between x and $yb \cdot \text{Let } x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	$\inf(tA) = \begin{cases} t \cdot \inf(A) \\ t \cdot \sup(A) \end{cases}$	<i>if t</i> > 0 <i>if t</i> < 0	
The set N is not bounded above. Proof Suppose that N is bounded above then N has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$ By example $6, \exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if N is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between x and $yb \cdot \text{Let } x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	Theorem 5		
Proof Suppose that N is bounded above then N has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$ By example $6, \exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Chen x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) Chen there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if N is unbounded in F. e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between x and $yb \cdot \text{Let } x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	The set $\mathbb N$ is not bounded	d above.	
Suppose that N is bounded above then N has a supremum (by complete axiom) say $u = \sup(\mathbb{N})$ By example 6, $\exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ (then x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) (then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property $f x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N} \rightarrow n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R} \text{ and } x < y$, then \exists infinitely irrationals between x and $yb \cdot \text{Let } x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	Proof		
complete axiom) say $u = \sup(\mathbb{N})$ By example 6, $\exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x'}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y $b \cdot \text{Let } x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Suppose that $\mathbb N$ is bound	led above then ℕ	l has a supremum (by
By example 6, $\exists n \in \mathbb{N} \ni u - 1 < n$ $\rightarrow u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \rightarrow C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \rightarrow \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \rightarrow C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}, \forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and $yb- Let x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	complete axiom) say u =	= sup(ℕ)	
→ $u < n + 1$ for this n. Since $n + 1 \in \mathbb{N} \to C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \to \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \to C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y b - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	By example 6, $\exists n \in \mathbb{N} \ni$	u - 1 < n	
Since $n + 1 \in \mathbb{N} \to C$! (for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \to \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \to C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and $yb- Let x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	$\rightarrow u < n+1$ for this n.		
(for $sup(\mathbb{N}) = u < n + 1 \in \mathbb{N} \to \mathbb{N}$ is bounded) Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then <i>x</i> is upper bound of $\mathbb{N} \to C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let <i>x</i> , <i>y</i> $\in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Since $n + 1 \in \mathbb{N} \to C!$		
Theorem 6 For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \to C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) $2 \cdot \mathbb{Q}$ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) $a \cdot \text{Let } x, y \in \mathbb{R} \text{ and } x < y$, then \exists infinitely rationals between x and $yb \cdot \text{Let } x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	$(\text{for } sup(\mathbb{N}) = u < n + 1)$	$1 \in \mathbb{N} \to \mathbb{N}$ is bo	ounded)
For every real number $x \exists n \in \mathbb{N}, n > x$. Proof Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then <i>x</i> is upper bound of $\mathbb{N} \to C$! (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Theorem 6	— — DI	
Proof Suppose not, <i>n</i> ≤ <i>x</i> , ∀ <i>n</i> ∈ N Then <i>x</i> is upper bound of N → <i>C</i> ! (theorem 5) Then there must exists <i>n</i> ∈ N ∋ <i>n</i> > <i>x</i> . Theorem 7 Archimedean property f <i>x</i> ∈ ℝ ⁺⁺ , then for any <i>y</i> ∈ ℝ, ∃ <i>n</i> ∈ N ∋ <i>nx</i> < <i>y</i> Proof If there is no <i>n</i> ∈ N ∋ <i>nx</i> < <i>y</i> , we have <i>nx</i> ≥ <i>y</i> , ∀ <i>n</i> ∈ N → <i>n</i> ≤ $\frac{y}{x}$, ∀ <i>n</i> ∈ N → $\frac{y}{x}$ is an upper bound for N → <i>C</i> ! (by Th. 5) Definition Let F be any field, F is called Archimedean field if N is unbounded in F. .e., ∀ <i>x</i> ∈ <i>F</i> , ∃ <i>n</i> ∈ N ∋ <i>n</i> > <i>x</i> Example 1- ℝ is arch. Field (by Th. 6) 2- ℚ is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let <i>x</i> , <i>y</i> ∈ ℝ and <i>x</i> < <i>y</i> , then ∃ infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let <i>x</i> , <i>y</i> ∈ ℝ and <i>x</i> < <i>y</i> , then ∃ infinitely irrationals between	For every real number x	$\exists n \in \mathbb{N}, n > x.$	
Suppose not, $n \le x, \forall n \in \mathbb{N}$ Then x is upper bound of $\mathbb{N} \to C!$ (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property $f x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}$, $\exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}, \forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and $yb- Let x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	Proof		
Then x is upper bound of $\mathbb{N} \to C$ (theorem 5) Then there must exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}$, $\exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\to n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let <i>x</i> , <i>y</i> $\in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Suppose not, $n \leq x, \forall n \in \mathbb{N}$		- 5)
Then there infust exists $n \in \mathbb{N} \ni n > x$. Theorem 7 Archimedean property $f x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}$, $\exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \to \frac{y}{x}$ is an upper bound for $\mathbb{N} \to C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and $yb- Let x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	Then there must exists $n \in \mathbb{C}$	\rightarrow C! (theorem	15)
f $x \in \mathbb{R}^{++}$, then for any $y \in \mathbb{R}$, $\exists n \in \mathbb{N} \ni nx < y$ Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C$! (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and $yb- Let x, y \in \mathbb{R} and x < y, then \exists infinitely irrationals between$	Then there must exists $n \in$ Theorem 7 Archimodoan n	$\mathbb{N} \supset \mathbb{N} > X$.	
Proof If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\Rightarrow n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) a - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y b - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	If $r \in \mathbb{R}^{++}$ then for any $v \in \mathbb{R}^{++}$	$\mathbb{R} \exists n \in \mathbb{N} \exists n$	$\zeta \leq 1$
If there is no $n \in \mathbb{N} \ni nx < y$, we have $nx \ge y$, $\forall n \in \mathbb{N}$ $\rightarrow n \le \frac{y}{x}$, $\forall n \in \mathbb{N} \rightarrow \frac{y}{x}$ is an upper bound for $\mathbb{N} \rightarrow C!$ (by Th. 5) Definition Let F be any field, F is called Archimedean field if \mathbb{N} is inbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let <i>x</i> , <i>y</i> $\in \mathbb{R}$ and <i>x</i> < <i>y</i> , then \exists infinitely irrationals between	Proof		<i>L</i> < <i>Y</i>
	If there is no $n \in \mathbb{N} \ni nx$.	< v , we have n	$x > y$, $\forall n \in \mathbb{N}$
Definition Let F be any field, F is called Archimedean field if N is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	$\rightarrow n < \frac{y}{2} \forall n \in \mathbb{N} \rightarrow \frac{y}{2}$	s an upper bour	and for $\mathbb{N} \to C!$ (by Th. 5)
Let F be any field, F is called Archimedean field if N is unbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Definition	b un upper bour	
Inbounded in F. .e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Let E be any field E is	called Archime	dean field if N is
inbounded in F. i.e., $\forall x \in F$, $\exists n \in \mathbb{N} \ni n > x$ Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	unbounded in F	s called Al chille	
Example 1- \mathbb{R} is arch. Field (by Th. 6) 2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Fheorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between <i>x</i> and <i>y</i> <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	ie $\forall x \in F \exists n \in \mathbb{N} \exists n > n$	Y	
2- \mathbb{Q} is arch. Field Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Example 1- \mathbb{R} is arch. Field	~ (by Th. 6)	
Exercise: prove that $sup(S) = \frac{1}{2}$ where $S = \{\frac{n-1}{2n}; n \in \mathbb{N}\}$ Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	$2-\mathbb{O}$ is arch. Field	(~,	
Theorem 8 (Density of real numbers) <i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	Evercise: $rrove that curr(C)$	$-\frac{1}{2}$ where $\varsigma -$	$\{\frac{n-1}{2}, n \in \mathbb{N}\}$
<i>a</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	The ensure \mathbf{O} (Densities f	$-\frac{2}{2}$	$2n$, $n \in \mathbb{N}$
<i>a</i> -Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely rationals between x and y <i>b</i> -Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	I neorem 8 (Density of real	al numbers)	nitoly notionala katuraa
<i>x</i> and <i>y</i> <i>b</i> - Let $x, y \in \mathbb{R}$ and $x < y$, then \exists infinitely irrationals between	u - Let $x, y \in \mathbb{K}$ and $x \in \mathbb{K}$	$< y$, then \exists infl	intery rationals between
u^{-} Let $x, y \in \mathbb{R}$ and $x \leq y$, then \exists minimizery intationals between	x and $yh_{-} Let x \in \mathbb{D} and x$	 y than ∃ infi 	nitaly irrationals batween
x and y	v - Let $x, y \in \mathbb{R}$ and x	$\langle y, uieii \exists iiiii$	intery in actorials between

University of Anbar		Third Year — First Semester
Department of Applied Mathematics	Lastura No. 2	Lectures in Mathematical analysis
	Lecture No. 5	
Proof :		
a) Since $x < y$ then $y - x$	x > 0	
by arch. Prop. $\exists n \in \mathbb{N}$	$n \ni n(y-x) > 1$	1
$y - x > \frac{1}{n}$ since r	$nx \text{ and } -nx \in \mathbb{R}$	
by theorem 6 \longrightarrow	$\exists m, m' \ni m > n$	x and $m' > -nx$
	m	' m
$\rightarrow -m$	$n < nx < m \rightarrow \frac{1}{n}$	$- < x < \frac{-}{n}$
since the set $\{-m', -n'\}$	$m' + 1,, m$ } is find	inite
\rightarrow let <i>m</i> ^{''} the smallest number	r such that $x \leq \frac{m}{n}$	<u>n</u>
$(x \leq \frac{m''}{n})$	يعة يحقق المتراجحة	(''m هي اصغر عدد في المجمو
Since $m'' - 1 < m'' \to \frac{m'' - 1}{r}$	ختيار " <i>m</i> (حتيار	(حسب ا
We have		
<i>m''</i>	$m^{\prime\prime}-1$ 1	1
$x < \frac{1}{n} =$	$=$ $\frac{n}{n}$ $+$ $\frac{1}{n} \leq 2$	$\alpha + \frac{1}{n}$
$\langle x + (y - x) = y \rightarrow r = \frac{m''}{n}$	is rational and <i>x</i>	< r < y
Continue in this way to find <i>n</i>	$r_i \in \mathbb{Q} \text{ and } x < r_i$	< r,
and $x <$	$\dots < r_2 < r_1 < r$	< <i>y</i>
Remark : Q is ordered field a	and arch. Prop, bu	ıt ${\mathbb Q}$ is not complete.
Theorem 9		
Q is not complete field.		
Proof		
Suppose Q is complete.		_
Let $S = \{x : x \in \mathbb{Q}^+ \text{ and } x^2 \le$	$2\} \subset [0, \sqrt{2}) \subset \mathbb{F}$	<u> </u>
$\rightarrow S \neq \phi$ since $1 \in S$ and S is	s bounded above	by $4 < 3 < 2 < \sqrt{2}$
By completeness , $\exists r \in \mathbb{Q} \ni$	$r = \sup(S)$, axio	om of Q
Suppose $T = \{x \in \mathbb{R}^+ : x^2 \le 2\}$	$\{2\} = [0, \sqrt{2}]$	
$\rightarrow S \subset T$ and $\sqrt{2}$ is irrational	upper bound of S	5
$\rightarrow r < \sqrt{2}$		
By Th.8, $\exists \ \bar{r} \in \mathbb{Q} \ \ni r < \bar{r} < \gamma$	$\sqrt{2}$	
$\rightarrow (\bar{r})^2 < 2 \rightarrow \bar{r} \in S \rightarrow C!$		
Since $r = \sup(S) < r^2 \rightarrow \mathbb{Q}$ if	is not complete.	
Theorem (10) (WITHOUT I	PROOF)	
The equation $x^2 = 2$ has a ur	nique positive sol	ution in R
	4	

University of Anbar		Third Year — First Semester
Department of Applied Mathematics		Lectures in Mathematical analysis
Collage of Science	Lecture No. 3	By: Dr. Rifaat Saad Abdul-Jabbar

References

1- Principles Of Mathematical Analysis - W.Rudin. <u>https://59clc.files.wordpress.com/2012/08/functional-analysis-_-</u> <u>rudin-2th.pdf</u>