University of Anbar

Collage of Science

Department of Applied Mathematics

Fourth Year – First Semester

Lectures in Functional analysis

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Lecture No. 1

Metric Space

Course name : Functional analysis I

Class : Fourth

Duration: 15 weeks - 3 hours per week.

1. Metric space
1.1 Examples of metric spaces.
1.2 Open sets and closed sets in metric spaces.
1.3 continuous mappings
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2. Vector spaces
2.1 Definitions and examples.
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3.1 Definitions and examples.
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3.4 Completeness with some related theorems.
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1 Metric Space

In calculus we study functions defined on the real line $\mathbb{R}$. A little reflection shows that in limit processes and many other considerations we use the fact that on R we have available a distance function, call it d, which associates a *distance* $d(x,y) = |x-y|$ with every pair of points $x, y \in \mathbb{R}$. In the plane and in "ordinary" three-dimensional space the situation is similar.

In functional analysis we shall study more general "spaces" and "functions" defined on them.

1.1-1 **Definition** (Metric space, metric). A *metric space* is a pair *(X, d)*, where X is a set and *d* is a *metric on* X (or ***distance function*** *on K),* that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(Ml) d is real valued, finite and nonnegative.

(M2) $d(x,y) = 0$ if and only If $x = y$

(M3) $d(x,y) = d(y,x)$ *(Symmetry).*

(M4) $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality).

Note The symbol × denotes the Cartesian product of sets: $A \times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence $X \times X$ is the set of all ordered pairs of elements of X.

Definition A **subspace** $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset

$Y \subset X$

and restrict d to $Y \times Y$; thus the metric on Y is the restriction:

$$\tilde{d} = d|_{Y \times Y}.$$

$\tilde{d}$ is called the metric **induced** on Y by d.

Examples

1. Real line $\mathbb{R}$ This is the set of all real numbers, taken with the usual metric defined by

$$d(x, y) = |x - y|$$

1.2 Euclidean plane

The metric space $\mathbb{R}^2$ called the *Euclidean plane*, is obtained if we take the set of ordered pairs of real numbers,

written $x = (\zeta_1, \zeta_2), y = (\eta_1, \eta_1)$, etc., and the *Euclidean metric* defined by

$$d(x, y) = \sqrt{(\zeta_1 - \eta_1)^2 + (\zeta_2 - \eta_2)^2}$$

Another metric space is obtained if we choose the same set as before but another metric d_1 defined by

$$d_1(x, y) = |\zeta_1 - \eta_1| + |\zeta_2 - \eta_2|$$

See figure 1, for illustration

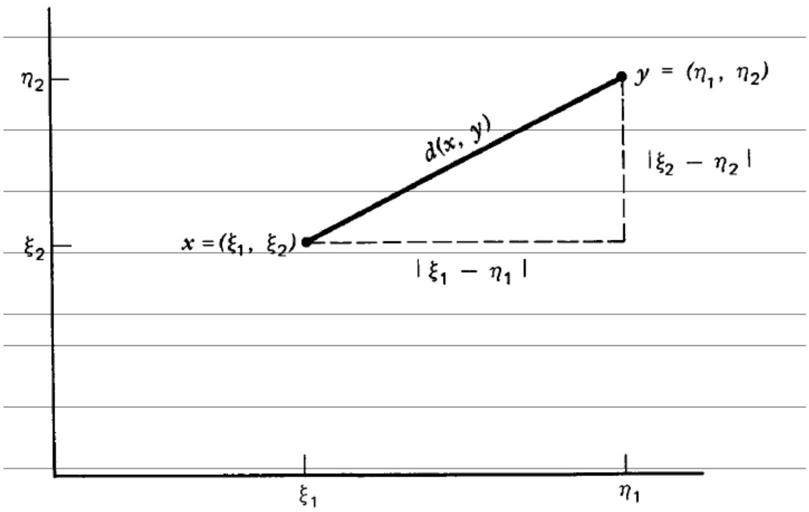

Figure 1: Euclidean metric on the plane

This illustrates the important fact that from a given set (having more than one element) we can obtain various metric spaces by choosing different metrics. (The metric space with metric $d_1$does not have a standard name.

3-Three-dimensional Euclidean space $\mathbb{R}^3$

This metric space consists of the set of ordered triples of real numbers $x = (\zeta_1, \zeta_2, \zeta_3), y = (\eta_1, \eta_2, \eta_3)$, etc., and the *Euclidean metric* defined by

$$d(x,y) = \sqrt{(\zeta_1 - \eta_1)^2 + (\zeta_2 - \eta_2)^2 + (\zeta_3 - \eta_3)^2}$$

4- **Euclidean space** $\mathbb{R}^n$**, unitary space** $\mathbb{C}^n$**, complex plane** $\mathbb{C}$. The previous examples are special cases of n-dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set of all ordered n-tuples of real numbers, written $x = (\zeta_1, \dots, \zeta_n), y = (\eta_1, \dots, \eta_n)$, etc. and the *Euclidean metric* defined by:

$$d(x,y) = \sqrt{(\zeta_1 - \eta_1)^2 + \cdots + (\zeta_n - \eta_n)^2}$$

n-dimensional unitary space $\mathbb{C}^n$ is the space of all ordered *n*-tuples of c*omplex* numbers with metric defined by:

$$d(x,y) = \sqrt{|\zeta_1 - \eta_1|^2 + \cdots + |\zeta_n - \eta_n|^2}$$

When $n = 1$ this is the *complex plane* C with the usual metric defined by

$$d(x,y) = |x - y|$$

(C" is sometimes called *complex Euclidean n-space.)*

5- Sequence space l^∞ , This example and the next one give a first impression of how surprisingly general the concept of a metric space is as a set X we take the set of all bounded sequences of complex numbers; that is, every element of X is a complex sequence

$x = (\zeta_1, \zeta_2, \dots)$ briefly $x = (\zeta_i)$ such that for all $j = 1, 2, \dots$ we have

$$|\zeta_i| \le c_x$$

where c_x is a real number which may depend on *x*, but does not depend on i.

We choose the metric defined by

$$d(x,y) = \sup_{j \in \mathbb{N}} |\zeta_j - \eta_j|$$

where $y = \eta_j \in X$and $N = \{l, 2, \dots\}$, and sup denotes the supremum (least upper bound). The metric space thus obtained is generally denoted by I^∞ .
I^∞ is a *sequence space* because each element
of X (each point of X) is a sequence.

6- **Function space $C[a, b]$.** As a set X we take the set of all real-valued functions x, y, ... which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$. Choosing the metric defined by

$d(x, y) = max|x(t) - y(t)|$,

where max denotes the maximum, we obtain a metric space which is denoted by $C[a, b]$. (The letter C suggests "continuous.") This is a *function space* because every point of $C[a, b]$is a function.

7- **Discrete metric space**. Vie take any set X and on it the so-called *discrete metric* for X, defined by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

This space *(X, d)* is called a *discrete metric space*. It rarely occurs in applications. However, we shall use it in examples for illustrating certain concepts .

Problems

1. Show that the real line is a metric space.
2. Does $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers?
3. Show that $d(x.y) = \sqrt{|x - y|}$ defines a metric on the set of all real numbers.
4. Find all metrics on a set X consisting of two points.
5. Let d be a metric on X. Determine all constants k such that (i) kd,
   (ii) $d + k$ is a metric on X.

References

1-Kreyszig, Erwin. **Introductory functional analysis with applications,1978.**
https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE15AF53_kov/Kreyszig%20-%20Introductory%20Functional%20Analysis%20with%20Applications%20(1).pdf

2- Functional Analysis, Second Edition, Walter Rudin,1991.
https://59clc.files.wordpress.com/2012/08/functional-analysis-_-rudin-2th.pdf