University of Anbar

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Department of Applied Mathematics

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Lectures in Functional analysis

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Lecture No. 2

Open Set, Closed Set, Neighborhood

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We first consider important types of subsets of a given metric space $X = (X, d)$.

Definition (Ball and sphere). Given a point $x_\circ \in X$and a real number $r > 0$, we **define three types of sets.**

a. $B(x_0;\ r) = \{x \in X : d(x, x_0) < r\}$ (Open ball)

b. $\widetilde{B}(x_0;\ r) = \{x \in X : d(x, x_0) \leq r\}$ (Close ball)

c. $S(x_0;\ r) = \{x \in X : d(x, x_0) = r\}$ (Shell)

In all three cases, x_0 is called the *center* and *r* the *radius.*

We see that an open ball of radius r is the set of all points in X whose distance from the center of the ball is less than *r.* Furthermore, the definition immediately implies that

$$S(x_0;\ r) = \tilde{B}(x_0;\ r) - B(x_0;\ r)$$

Warning. In working with metric spaces, it is a great advantage that we use a terminology which is analogous to that of Euclidean geometry. However, we should beware of a danger, namely, of assuming that balls and spheres in an arbitrary metric space enjoy the same properties as balls and spheres in $\mathbb{R}^3$. This is not so. An unusual property is that a sphere can be empty. For example, in a discrete metric space we have $S(x_0;\ r) = \phi \quad 0$ if $r \neq 1$. (What about spheres of radius 1 in this case?) .

Let us proceed to the next two concepts, which are related.

Definition (Open set, closed set). A subset M of a metric space X is said to be *open* If It contains a ball about each of its points. A subset K of X is said to be *closed* if its complement (in X) is open, that is, $K^c = X - K$ is open.

The reader will easily see from this definition that an open ball is an open set and a closed ball is a closed set.

An open ball $B\ (x_0;\ \epsilon)$ of radius ϵ is often called an ϵ*-neighborhood* of x_0. By a neighborhood of x_0 we mean any subset of X which contains an ϵ-neighborhood of x_0

We see directly from the definition that every neighborhood of x_0 contains x_0; in other words, x_0 is a point of each of its neighborhoods. And if N is a neighborhood of x_0 and $N \subset$ M, then M is also a neighborhood of x_0.

We call x_0 an interior point of a set $M \subset X$ if M is a neighborhood of xo. The interior of M is the set of all interior points of M and may be denoted by M^0or $Int\ (M)$, but there is no generally accepted notation. $lnt\ (M)$ is open and is the largest open set contained in M

It is not difficult to show that the collection of all open subsets of X, call it $\mathbb{T}$, has the following properties:

(T1) $\phi \in \mathbb{T}$, $X \in \mathbb{T}$

(T2) The union of any members of $\mathbb{T}$ is a member of $\mathbb{T}$.

(T3) The intersection of finitely many members of $\mathbb{T}$ is a member of $\mathbb{T}$.

Proof. (T1) follows by noting that ϕ is open since ϕ has no elements and, obviously, X is open. We prove (T2). Any point x of the union U of open sets belongs to (at least) one of these sets, call it M, and M contains a ball B about x since M is open. Then $B \subset U$, by the definition of a union. This proves (T2). Finally, if y is any point of the intersection of open sets $M_{1,}, \dots M_n$, then each M_i contains a ball about y and a smallest of these balls is contained in that intersection. This proves (T3).

We mention that the properties (T1) to (T3) are so fundamental that one wants to retain them in a more general setting. Accordingly, one defines a topological space $(X, \mathbb{T})$ to be a set X and a collection $\mathbb{T}$ of subsets of X such that $\mathbb{T}$ satisfies the *axioms* (T1) to (T3). The set $\mathbb{T}$ is called *a topology for* X. From this definition we have: ***A metric space is a topological space***

Open sets also play a role in connection with continuous mappings, where continuity is a natural generalization of the continuity known from calculus and is defined as follows.

Definition (**Continuous mapping**). Let $X = (X, d)$ and $Y = (Y, \bar{d})$ be metric spaces. A mapping $T: X \rightarrow Y$ is said to be *continuous at a point* $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\bar{d}(Tx, Txo) < \epsilon \qquad \text{for all } x \text{ satisfying } d(x, xo) < \delta.$$

T is said to be *continuous* if it is continuous at every point of X.

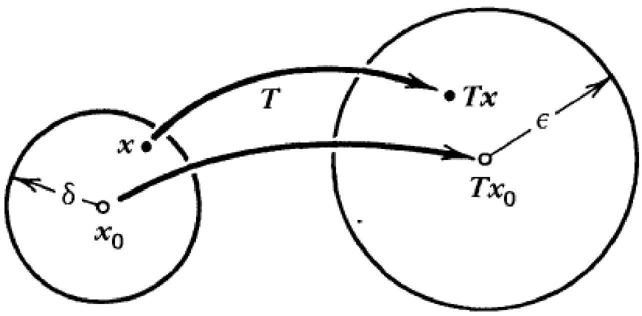

Figure 2. Illustration of definition of continuous function in case of Euclidian space $Y = \mathbb{R}^2$

Theorem (Continuous mapping). *A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

Proof. *(a)* Suppose that T is continuous. Let $S \subset Y$ be open and S_o the inverse image of S. If $S_0 = \phi$, it is open. Let $S_0 \neq \phi$. For any $x_0 \in S_0$ let $y_0 = Tx_0$; Since S is open, it contains an ϵ-neighborhood N of y_0. Since T is continuous, x_0 has a δ-neighborhood N_0, which is mapped into N. Since $N \subset S$, we have $N_0 \subset S_0$, so that S_0 is open because $x_0 \in S_0$ was arbitrary.

(b) Conversely, assume that the inverse image of every open set in Y is an open set in X. Then for every $x_0 \in X$ and any ϵ -neighborhood N of Tx_0-; the inverse image N_0 of N is open, since N is open, and N_0 contains x_0. Hence N_0 also contains a δ-neighborhood of x_0., which is mapped into N because N_0. is mapped into N. Consequently, by the definition, T is continuous at x_0. Since $x_0 \in X$ was arbitrary, T is continuous.

We shall now introduce two more concepts, which are related. Let M be a subset of a metric space X. Then a point x_0. of X (which mayor may not be a point of M) is called an accumulation point *of* M (or *limit point of* M) if every neighborhood of x_0 contains at least one point $Y \in M$ distinct from x_0. The set consisting of the points of M and the accumulation points of M is called the closure *of* M and is denoted by $\bar{M}$.
It is the smallest closed set containing M.

Before we go on, we mention another unusual property of balls in a metric space. Whereas in $\mathbb{R}^3$ the closure $\overline{B(x_0;\ r)}$ of an open ball $B(x_0;\ r)$ is the closed ball $\tilde{B}(x_0;\ r)$, this may not hold in a general metric space.
Using the concept of the closure, let us give a definition which will be of particular importance in our further work:

Definition (**Dense set, separable space**). A subset M of a metric space X is said to be *dense in* X if

$$\bar{M} = X.$$

X is said to be *separable* if it has a countable subset which is dense in X.

Examples

1. Real line $\mathbb{R}$. *is separable.*
   Proof. The set $\mathbb{Q}$ of all rational numbers IS countable and is dense in $\mathbb{R}$.
2. Complex plane $\mathbb{C}$. *The complex plane $\mathbb{C}$ is separable.*
   Proof. A countable dense subset of $\mathbb{C}$ is the set of all complex numbers whose real and imaginary parts are both rational.

3. Discrete metric space. *A discrete metric space* X *is separable if and only if* X *is countable.*

Proof The kind of metric implies that no proper subset of X can be dense in X. Hence the only dense set in X is X itself, and the statement follows.

4. *The* Space l^∞ *is not separable.*

Proof. Let $y = (\eta_1, \eta_2, \eta_3, \dots)$ be a sequence of zeros and ones. Then $y \in l^\infty$. With y we associate the real number $\hat{y}$ whose binary representation is

$$\frac{\eta_1}{2^1} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + \cdots.$$

We now use the facts that the set of points in the interval [0,1] is uncountable, each $\hat{y} \in [0,1]$ has a binary representation, and different $\hat{y}$'s have different binary representations. Hence there are uncountably many sequences of zeros and ones. The metric on l^∞ shows that any two of them which are not equal must be of distance 1 apart. If we let each of these sequences be the center of a small ball, say, of radius 1/3, these balls do not intersect and we have uncountably many of them. If M is any dense set in l^∞, each of these nonintersecting balls must contain an element of M. Hence M cannot be countable. Since M was an arbitrary dense set, this shows that l^∞ cannot have dense subsets which are countable. Consequently, l^∞ is not separable.

Problems

1. If x_0 is an accumulation point of a set A ⊂ (X, d), show that any neighborhood of x_0 contains infinitely many points of A.

2. (Continuous mapping) Show that a mapping $T: X \to Y$ is continuous if and only if the inverse image of any closed set $M \subset Y$ is a closed set in X.

3. Show that the image of an open set under a continuous mapping need not be open.

References

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