University of Anbar

Collage of Science

Department of Applied Mathematics

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Lectures in Functional analysis

By: Dr. Lecturer Rifaat Saad Abdul-Jabbar

Lecture No. 3

Convergence., Cauchy Sequence., Completeness

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Definition (Convergence of a sequence, limit). A sequence (x_n) in a metric space X = *(X, d)* is said to *converge* or to *be convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n - x) = 0$$

x is called the *limit* of (x_n) and we write

$$\lim_{n\to\infty} (x_n) = x$$

or simply:

$$x_n \to x$$

We say that (x_n) converges to x or has the limit x. If (x_n) is not convergent, it is said to be divergent.

Definition

We call a nonempty subset $M \in X$ a *bounded set* if its *diameter:*

$$\delta(M) = sup\, d(x, y), \qquad x, y \in M$$

is finite. And we call a sequence x_n in X a bounded sequence If the corresponding point set is a bounded subset of X.

Lemma (Boundedness, limit).

Let X = (X, d) be a metric space. Then:

(a) A convergent sequence in is bounded and its limit is unique.

(b) If $x_n \to x$ *and* $y_n \to y$ *in X, then* $d(x_n, y_n) \to d(x, y)$.

Proof. *(a)* Suppose that $x_n \to x$. Then, taking *e = 1*, we can find an N such that d $(x_n, x) < 1$ for all n > N. Hence by the triangle inequality (M4), for all n we have $d(x_n, x) < 1 + a$ where

$$a = max\,\{d(x_1, x), \dots, d(x_N, x)\}.$$

This shows that (x_n) is bounded.

Assuming that $x_n \to x$ and $x_n \to z$, we obtain from (M4)

$$0 \le d(x,z) \le d(x,x_n) + d(x_n\,,z) \to 0+0$$

and the uniqueness $x = z$ of the limit follows from (M2).

(b) We know that

$$d(x_n, y_n) \le d(x_n\,,x) + d(x,y) + d(y, y_n)$$

Then we obtain

$$d(x_n, y_n) - d(x,y) \le d(x_n\,,x) + d(y_n, y)$$

and a similar inequality by interchanging x_n and x and y_n and y and multiplying by -1. Together,

$$\left|d(x_n, y_n) - d(x,y)\right| \le d(x_n\,,x) + d(y_n, y) \to 0 \quad \text{as } n \to \infty$$

Definition (Cauchy sequence, completeness). A sequence (x_n) in a metric space X = (X, d) is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_n, x_m) < \epsilon \qquad \forall m, n > N$$

The space X is said to be ***complete*** if every Cauchy sequence in X converges in X.

Theorem (Real line). *The real line is complete metric spaces.*

Example

Let $X = (0,1]$, with the usual metric defined by $d(x,y) = |x-y|$, and the sequence (x_n), where $x_n = 1/n$ and $n = 1,2,\ldots$ this is a Cauchy sequence, but it does not converge, because the point 0 is not a point of X.

Theorem (Convergent sequence). *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof

If $x_n \to x$, then for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_n, x) < \frac{\epsilon}{2} \quad , \qquad \forall n > N$$

Hence by the triangle inequality we obtain for m, n > N

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that (x_n) is Cauchy.

Theorem (Closure, closed set)

Let M be a nonempty subset of *a metric space (X, d) and* $\bar{M}$ *its closure Then*

(a) $x \in \bar{M}$ *if and only if there is a sequence* (x_n) *in* M*such that*

$x_n \to x$,

(b) M *is closed if and only if the situation* $x_n \in M, x_n \to x$, *implies*

That $x \in M$.

Proof.

(a) Let $x \in \bar{M}$.

If $x \in M$, a sequence of that type is $(x, x, \ldots)$.

If x $\notin$ M, it is a point of accumulation of M.

Hence for each $n = 1, 2, \ldots$ the ball $B(x: 1/n)$ contains an $x_n \in M$,

and $x_n \to x$ because $1/n \to 0$ as $n \to \infty$.

Conversely, if (x_n) is in M and $x_n \to x$,

then $x \in M$ for every neighborhood of x contains points $x_n \neq x$,

so that x is a point of accumulation of M.

Hence $x \in \bar{M}$, by the definition of the closure.

(b) M is closed if and only if $M = \bar{M}$, so that *(b)* follows directly from *(a)*

Theorem (Complete subspace). A *subspace* M *of a complete metric space* X *is itself complete if and only if the set* M *is closed in* X**.**

Proof. Let M be complete. Then, for every $x \in M$ there is a sequence (x_n) in M which converges to x. Since (x_n) is Cauchy and M is complete, (*Every convergent sequence in a metric space is a Cauchy sequence.*)

(x_n) converges in M, the limit being unique.

Hence $x \in M$. This proves that M is closed because $x \in M$ was arbitrary.

Conversely, let M be closed and (x_n) Cauchy in M.

The $x_n \to x \in X$ n, which implies $x \in M$ by *above theorem (a)*, and $x \in M$ since $M = \bar{M}$ by assumption.

Hence the arbitrary Cauchy sequence (x_n) converges in M, which proves completeness of M.

Theorem (Continuous mapping). *A mapping $T: X \to Y$ of a metric space (X, d) into a metric space $(Y, \bar{d})$ is continuous at a point* $x_n \in X$ *if and only if*

$$x_n \to x_0 \quad implies \quad Tx_n \to Tx_0$$

Proof

Let T be continuous at x_0 then, for a given $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$d(x, x_0) < \delta \quad implies \quad \bar{d}(Tx, Tx_0) < \epsilon$$

Let $x_n \to x_0$ then there is an N such that for all $n > N$ we have $d(x_n, x_0) < \delta$

Hence for all $n > N$,

$$\bar{d}(Tx_n, Tx_0) < \epsilon$$

By definition this means that $Tx_n \to Tx_0$

Conversely, we assume that

$$x_n \to x_0 \quad implies \quad Tx_n \to Tx_0$$

and prove that then T is continuous at x_0. Suppose this is false. Then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is an $x \neq x_0$ satisfying

$$d(x, x_0) < \delta \text{ but } \bar{d}(Tx, Tx_0) \geq \epsilon$$

In particular, for $\delta = 1/n$ there is an x_n satisfying

$$d(x_n, x_0) < 1/n \text{ but } \bar{d}(Tx_n, Tx_0) \geq \epsilon$$

Clearly $x_n \to x_0$ but (Tx_n) does not converge to Tx_0. This contradicts $Tx_n \to Tx_0$ and proves the theorem.

References

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