

Chapter Two

Classification of Partial Differential Equations

Since the solution procedure of a partial differential equation (PDE) depends on the type of the equation, it is important to study various classifications of PDEs. Imposition of initial and/or boundary conditions also depends on the type of PDE. Most of the governing equations of fluid mechanics and heat transfer are expressed as second-order PDEs and therefore classification of such equations is considered.

2-1 Linear and Nonlinear PDEs:

In a linear PDE, the dependent variable and its derivatives enter the equation linearly, i.e., there is no product of the dependent variable or its derivatives. An example of a linear PDE is the one-dimensional wave equation

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x},$$

Where a is the speed of sound which is assumed constant.

On the other hand, a nonlinear PDE contains a product of the dependent variable and/or a product of its derivatives. An example of a nonlinear PDE is the inviscid Burgers equation:

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}$$

If a PDE is linear in its highest order derivatives, it is called a quasi-linear PDE.

2-2 Second-Order PDEs:

To classify the second-order PDE, consider the following equation

$$A \frac{\partial^2 \phi}{\partial x^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \frac{\partial \phi}{\partial x} + E \frac{\partial \phi}{\partial y} + F \phi + G = 0$$

Where, in general, the coefficients $A, B, C, D, E, F,$ and G are functions of the independent variable x and y and the dependent variable ϕ .

The equation will be

(a) elliptic if $B^2 - 4AC < 0$

(b) parabolic if $B^2 - 4AC = 0$, or

(c) hyperbolic if $B^2 - 4AC > 0$

Note that the classification depends only on the coefficients of the highest order derivatives

2-3 Elliptic Equations:

A partial differential equation is elliptic in a region if $(B^2 - 4AC) < 0$ at all points of the region. A disturbance is propagated instantly in all directions with region. Examples of elliptic equations are Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

and Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$

The domain of solution for an elliptic PDE is a

closed region, R , shown in Figure (1).

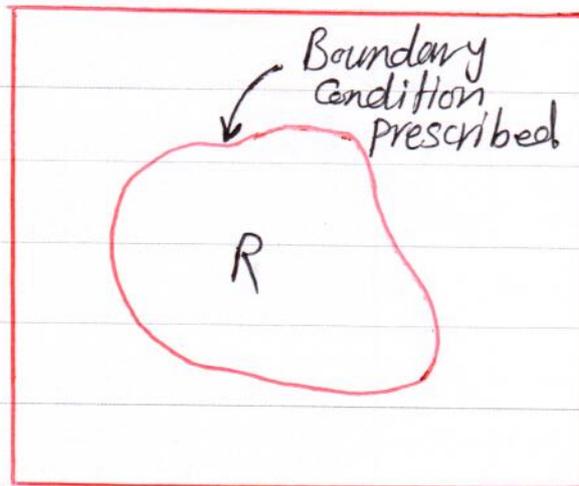


Figure 1. The domain of solution for an elliptic PDE.

On the closed boundary of R , either the value of the dependent variable, its normal gradient, or a linear combination of the two is prescribed.

2-4. Parabolic Equation:

A partial differential equation is classified as parabolic if $(B^2 - 4AC) = 0$ at all points of the region. The solution domain for a parabolic PDE is an open region, as shown in Figure (2).

Unsteady heat conduction in one dimension

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

and diffusion of viscosity, expressed as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

are examples of parabolic PDEs. An initial distribution of the dependent variable and two sets of boundary conditions are required for a complete description of the problem. The boundary conditions are prescribed as the value of the dependent variable or its normal derivative or a linear combination of the two. The solution of the parabolic equation marches downstream within the domain from the initial plane of data satisfying the specified boundary conditions.

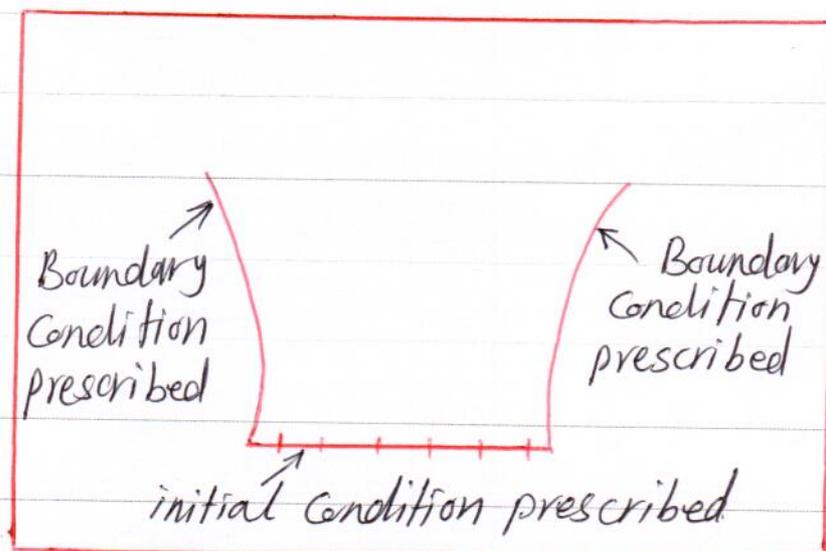


Figure (2) The domain of solution for a parabolic PDE.

2-5 Hyperbolic Equation:

A partial differential equation is called hyperbolic if $(B^2 - 4AC) > 0$ at all points of the region. An example of a hyperbolic equation is the second-order wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2}$$

A complete description of the flow governed by a second-order hyperbolic PDE requires two sets of initial conditions and two sets of boundary conditions. The initial conditions at $t=0$ may be expressed as

$$\phi(x, 0) = f(x)$$

and
$$\phi_t(x, 0) = g(x)$$

Where the functions f and g are specified for a particular problem.

For a first-order hyperbolic equation, such as

$$\frac{\partial \phi}{\partial t} = -a \frac{\partial \phi}{\partial x}$$

only one initial condition needs to be specified

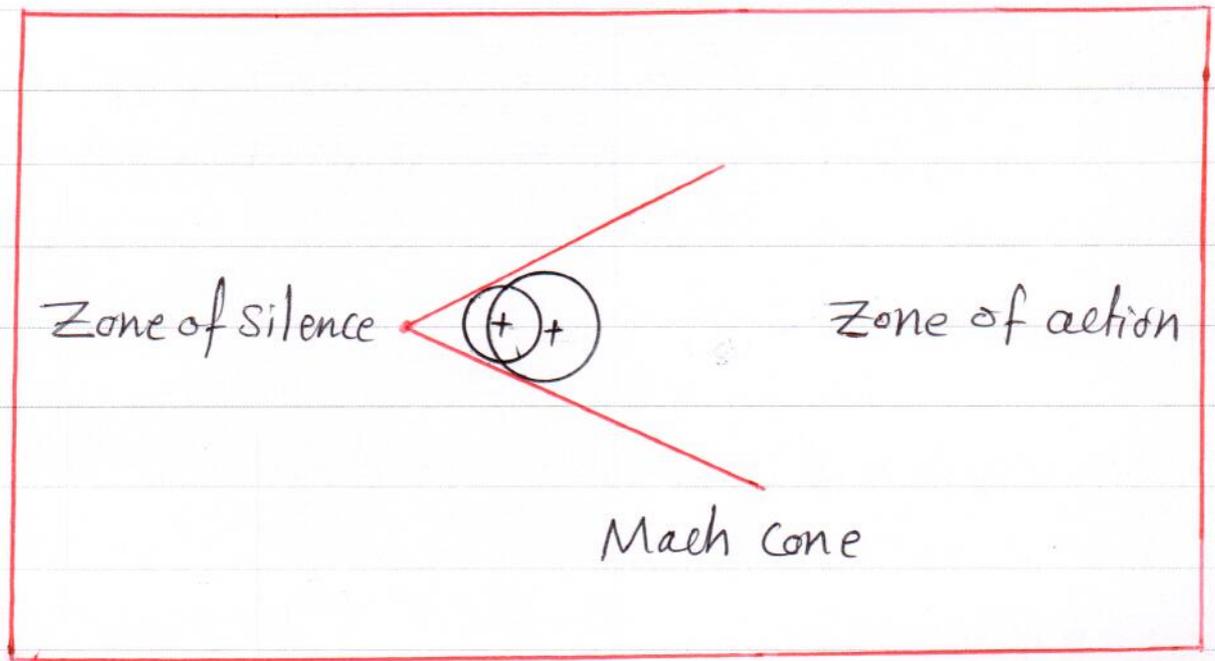


Figure (3) Propagation of disturbance in supersonic flow.

2-6 Initial and Boundary Conditions:

In order to obtain a unique solution of a PDE, a set of supplementary conditions must be provided to determine the arbitrary functions which result from the integration of the PDE. An initial condition is a requirement for which

the dependent variable is specified at some initial state.

A boundary condition is a requirement that the dependent variable or its derivative must satisfy on the boundary of the domain of the PDE.

Various types of boundary conditions which will be encountered are:

1. The Dirichlet boundary condition: If the dependent variable along the boundary is prescribed, it is known as the Dirichlet type.
2. The Neumann boundary condition: If the normal gradient of the dependent variable along the boundary is specified, it is called the Neumann type.
3. The Robin boundary condition: If the imposed boundary condition is a linear combination of the Dirichlet and Neumann types, it is known as the Robin type.
4. The Mixed boundary condition: Frequently the boundary condition along a certain portion

of the boundary is the Dirichlet type and, on another portion of the boundary, a Neumann type. This type is known as a mixed boundary condition.

As an example, consider transient conduction in two-space dimensions. Assume that a long rectangular bar has been heated to a temperature distribution of $T = f(x, y)$. An initial condition would then be prescribed such that

$$\text{for } t=0, \quad T = f(x, y)$$

Now, place the bar in an environment in which the lower and right sides are in contact with a convecting fluid of temperature T_f and a constant convection heat transfer coefficient of h , while the left side is insulated (adiabatic) and the upper side is kept at a constant temperature.

The corresponding boundary conditions are:

$$t \geq 0 \quad x=0, \quad \frac{\partial T}{\partial x} = 0$$

$$x=L, \quad \frac{\partial T}{\partial x} = -\frac{h}{k} (T - T_f)$$

$$y=0, \quad \frac{\partial T}{\partial y} = \frac{h}{k} (T - T_f)$$

$$y=H, \quad T = T_c$$

These are shown in Figure (4)-

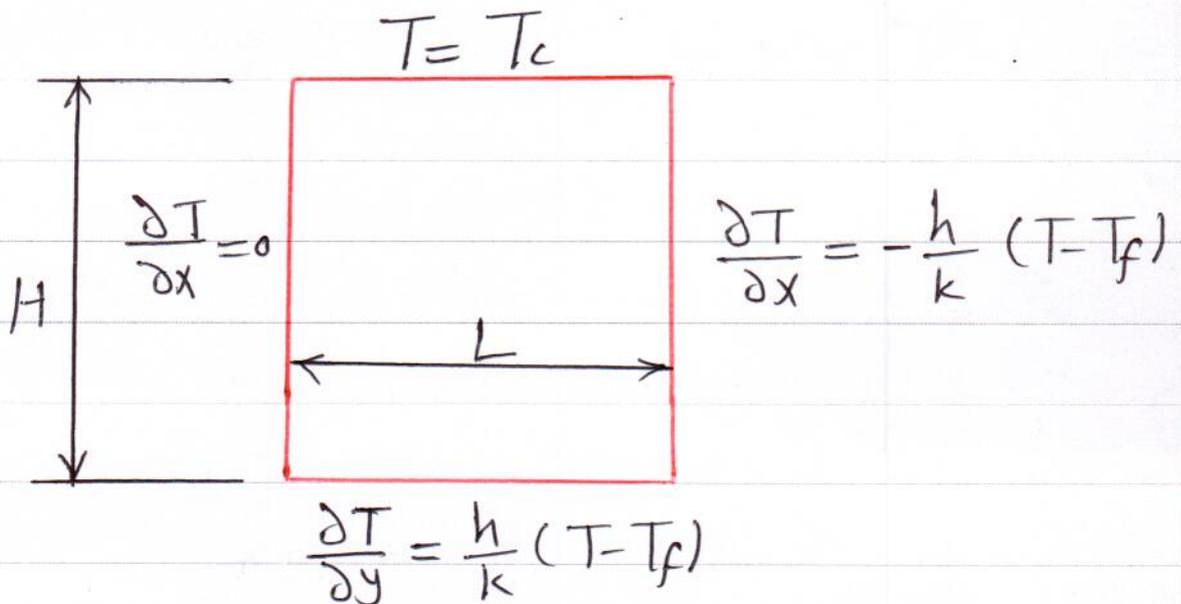


Figure (4). Sketch illustrating the imposed boundary conditions on the rectangular bar.