

$$\begin{aligned}
 a_{33} u_{43} + b_{33} u_{23} + c_{33} u_{33} + d_{33} u_{32} + e_{33} u_{34} &= f_{33} \\
 a_{34} u_{44} + b_{34} u_{24} + c_{34} u_{34} + d_{34} u_{33} &= f_{34} - e_{34} u_{35} \\
 b_{42} u_{32} + c_{42} u_{42} + e_{42} u_{43} &= f_{42} - a_{42} u_{52} - d_{42} u_{41} \\
 b_{43} u_{33} + c_{43} u_{43} + d_{43} u_{42} + e_{43} u_{44} &= f_{43} - a_{43} u_{53} \\
 b_{44} u_{34} + c_{44} u_{44} + d_{44} u_{43} &= f_{44} - a_{44} u_{54} - e_{44} u_{45}
 \end{aligned}$$

Where all the known quantities from the imposed boundary conditions have been moved to the right hand side and added to the known quantities from the previous n time level. The set of equations can be written in a matrix form as

$$\left[ \begin{array}{cccccc|c}
 c_{22} & e_{22} & 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 d_{23} & c_{23} & e_{23} & 0 & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & d_{24} & c_{24} & 0 & 0 & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 b_{32} & 0 & 0 & c_{32} & e_{32} & 0 & a_{32} & 0 & q_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & b_{33} & 0 & d_{33} & c_{33} & e_{33} & 0 & a_{33} & 0 & q_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & b_{34} & 0 & d_{34} & c_{34} & e_{34} & 0 & 0 & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & b_{42} & 0 & 0 & c_{42} & e_{42} & 0 & 0 & a_{42} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & b_{43} & 0 & d_{43} & c_{43} & e_{43} & 0 & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & b_{44} & 0 & d_{44} & c_{44} & e_{44} & 0 & 0 & a_{44} & 0 & 0 & 0 & 0
 \end{array} \right] \left[ \begin{array}{c}
 u_{22} \\ u_{23} \\ u_{24} \\ u_{32} \\ u_{33} \\ u_{34} \\ u_{42} \\ u_{43} \\ u_{44}
 \end{array} \right] = \left[ \begin{array}{c}
 f_{22} - b_{22} u_{12} - d_{22} u_{21} \\ f_{23} - b_{23} u_{13} \\ f_{24} - b_{24} u_{14} - e_{24} u_{25} \\ f_{32} - d_{32} u_{31} \\ f_{33} \\ f_{34} - e_{34} u_{35} \\ f_{42} - a_{42} u_{52} - d_{42} u_{41} \\ f_{43} - a_{43} u_{53} \\ f_{44} - a_{44} u_{54} - e_{44} u_{45}
 \end{array} \right]$$

The coefficient matrix is pentadiagonal. The solution procedure for a pentadiagonal system of equations is also very time-consuming. One way to overcome the shortcomings and inefficiency of the method described above is to use a splitting method. This method is known as the alternating direction implicit method or ADI. The algorithm produces two sets of tridiagonal simultaneous equations to be solved in sequence. The finite difference equations of model Eq(4-23) in the ADI formulation are

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^n}{(\Delta t)} = \alpha \left[ \frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2} \right] \quad (4-28a)$$

and

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^n}{(\Delta t)} = \alpha \left[ \frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{(\Delta y)^2} \right] \quad (4-28b)$$

The method is of order  $[\Delta x^2, \Delta y^2, \Delta t^2]$  and is unconditionally stable. Eqs (4-28a) and (4-28b) are written in the tridiagonal form as

$$-d_1 u_{i-1,j}^{n+\frac{1}{2}} + (1+2d_1) u_{i,j}^{n+\frac{1}{2}} - d_1 u_{i+1,j}^{n+\frac{1}{2}} = d_2 u_{i,j+1}^n + (1-2d_2) u_{i,j}^n$$

$$+ d_2 u_{i,j-1}^{n+1} \quad (4-29a)$$

and

$$-d_2 u_{i,j-1}^{n+1} + (1 + 2d_2) u_{i,j}^{n+1} - d_2 u_{i,j+1}^{n+1} = d_1 u_{i+1,j}^{n+\frac{1}{2}} \\ + (1 - 2d_1) u_{i,j}^{n+\frac{1}{2}} + d_1 u_{i-1,j}^{n+\frac{1}{2}} \quad (4-29b)$$

Where

$$d_1 = \frac{1}{2} dx = \frac{1}{2} \frac{\alpha \Delta t}{(\Delta x)^2}$$

and

$$d_2 = \frac{1}{2} dy = \frac{1}{2} \frac{\alpha \Delta t}{(\Delta y)^2}$$

The solution procedure starts with the solution of the tridiagonal system Eq (4-29a). The formulation of Eq(4-29a) is implicit in the x-direction and explicit in the y-direction, thus the solution at this stage is referred to as the x-sweep. Solving the tridiagonal system of (4-29a) provides the necessary data for the right-hand side of Eq(4-29b) to solve the tridiagonal system of (4-29b). In this equation, the FDE is implicit in

the  $y$ -direction and explicit in the  $x$ -direction, and it is referred to as the  $y$  sweep. Graphical presentation of the method is shown in Fig(4-6).

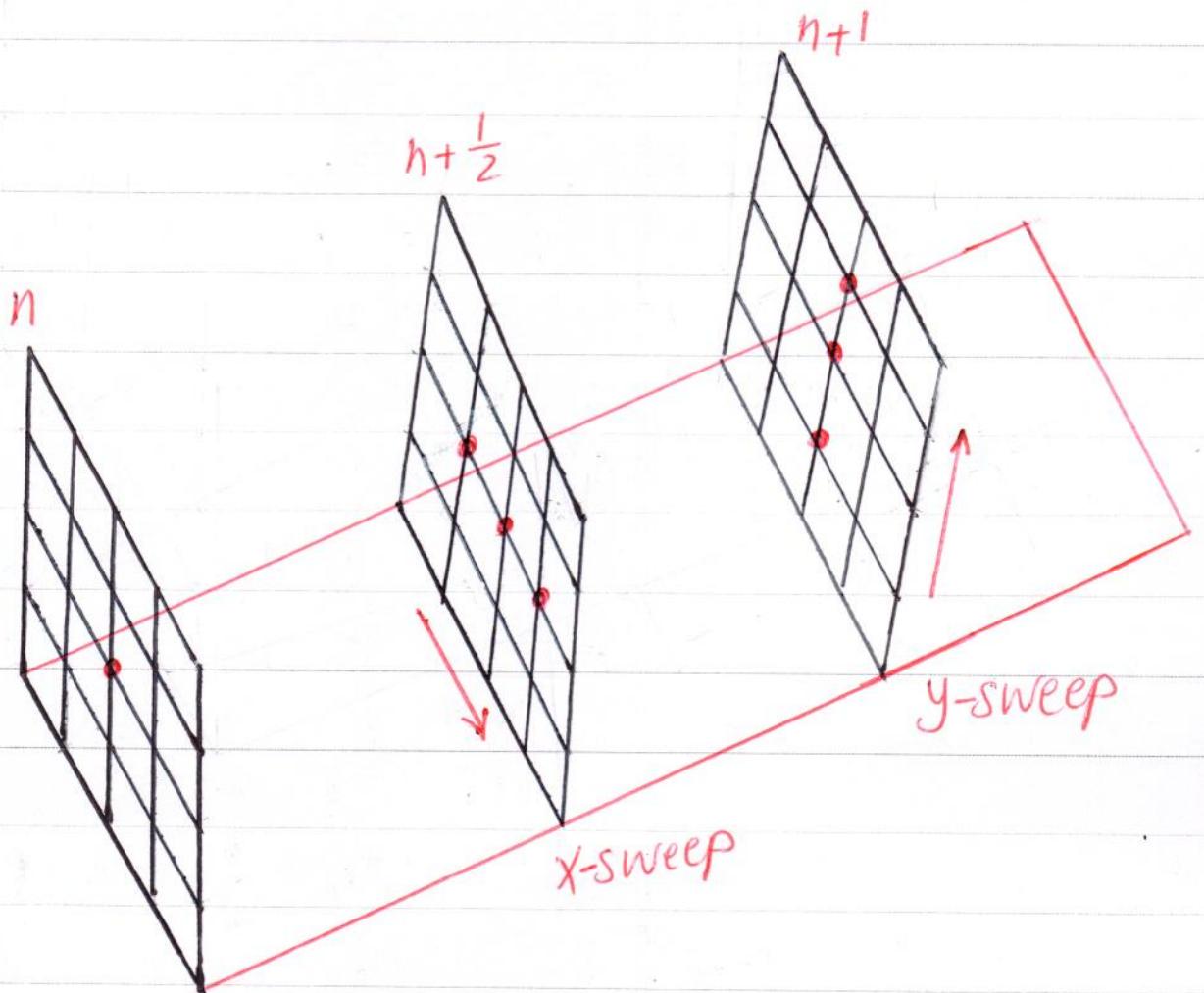


Fig (4-6) Illustration of the grid system for the ADI method.

### Example 4-1:

A copper plate of 1m long and infinite in other directions has an initial uniform temperature ( $T_i$ ) of 500K. The surface temperature ( $T_s$ ) at the two sides are suddenly decreased and maintained at 300K. The diffusivity of copper is  $117 \times 10^{-6}$  m<sup>2</sup>/s. Write computer programme to evaluate the temperature distribution within the plate as a function of time. The governing equation to be solved is the unsteady one-space dimensional heat conduction, which in Cartesian coordinates is

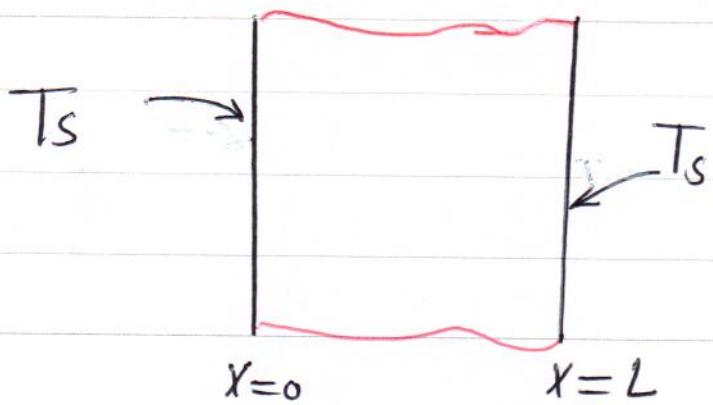
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Use the following techniques to solve the problem.

- (a) FTCS explicit
- (b) DuFort-Frankel
- (c) Laasonen
- (d) Crank-Nicolson

Use  $\Delta x = 0.05$  and  $\Delta t = 0.01$  for each method. In all cases the solution is to be printed and

plotted for all  $x$  locations for the time intervals from 0.0 to 500 sec.



Solution:

(a)

$$\frac{\alpha \Delta t}{(\Delta x)^2} = \frac{117 \times 10^{-6} (0.01)}{(0.05)^2} = 4.68 \times 10^{-4} \sqrt{\frac{1}{2}}$$

Since  $\frac{\alpha \Delta t}{\Delta x^2} \sqrt{\frac{1}{2}} > 1$ , FTCS explicit method

is stable.

### Example 4-2:

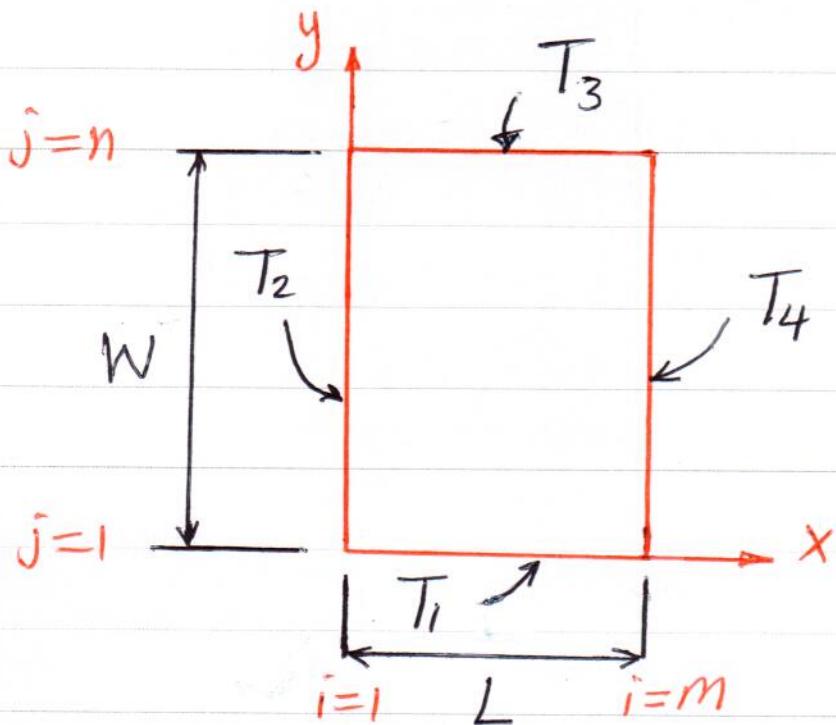
A long bar with rectangular cross-section, as shown in figure. The bar is initially at temperature of  $T_0$ . Subsequently, its surfaces are subjected to the constant temperatures of  $T_1$ ,  $T_2$ ,  $T_3$  and  $T_4$ , as depicted in figure. It is required to compute the transient solution where the governing equation is

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

The bar is composed of copper with a thermal conductivity of  $380 \text{ W/m}\cdot\text{C}$  and a thermal diffusivity of  $11.234 \times 10^{-5} \text{ m}^2/\text{sec}$ , both assumed constant for this problem. The rectangular bar has dimensions of  $L=0.3\text{m}$ , and  $W=0.4\text{m}$ . The computational grid is specified by  $m=31$  and  $n=41$ .

Use the FTCS explicit and ADI methods with time steps of  $0.02$  and  $1.0 \text{ sec}$  to compute the transient solution. The initial and boundary conditions are specified as:  $T_0=0^\circ\text{C}$ ,  $T_1=40^\circ\text{C}$

$T_2 = 0^\circ\text{C}$ ,  $T_3 = 10^\circ\text{C}$  and  $T_4 = 0^\circ\text{C}$ .



Solution:

(a) FTCS method

$$\Delta x = \frac{L}{(M-1)} = \frac{0.3}{(31-1)} = 0.01$$

$$\Delta y = \frac{W}{(n-1)} = \frac{0.4}{(41-1)} = 0.01$$

$$\therefore \alpha \Delta t = \frac{\alpha \Delta t}{\Delta x^2} = \frac{(11.234 \times 10^{-5})(0.02)}{(0.01)^2} = 0.0224$$

Similarly,

$$\alpha \Delta t = 0.0224$$

The stability condition is

$$dx + dy = 0.0224 + 0.0224 = 0.0448 \leq \frac{1}{2}$$

$$\frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = \alpha \left[ \frac{T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n}{(\Delta x)^2} + \frac{T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n}{(\Delta y)^2} \right]$$

Rearranging above equation gives

$$T_{i,j}^{n+1} = dx T_{i+1,j}^n + dx T_{i-1,j}^n + (1 - 2dx - 2dy) T_{i,j}^n + dy T_{i,j+1}^n + dy T_{i,j-1}^n$$

Where

$$dx = \frac{\alpha \Delta t}{\Delta x^2}$$

$$dy = \frac{\alpha \Delta t}{\Delta y^2}$$